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Mathematical Concept Representations in the Early Years of School: Analyzing Student's Errors

Representações de Conceitos Matemáticos nos Primeiros Anos do Ensino Escolar: Análise dos Erros dos Alunos

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Keywords: Math errors. Mathematical representations. Error analysis. Mathematics education. Early years.

Abstract: In this paper we analyze students' mathematics errors in the early years of Brazilian school. To achieve this goal, we developed a questionnaire with eight math problems that included basic operations. This instrument was distributed to 76 third-grade students in three different classes from three different schools. To comprehend what caused the errors, we analyzed each question. To perform these analyzes we used the following categories: Errors due to language difficulties; Errors due to difficulties in obtaining spatial information; Errors due to incorrect associations or rigidity of thinking; Errors due to the application of irrelevant rules or strategies; Errors due to deficient mastery of prerequisite skills, facts, and concepts; Incorrect reproduction of the task; Error of counting; Errors in the assemblage of the arithmetic operations; Errors in the summing up with values bigger than ten; and Specific errors of subtraction. We believe that using errors as motivation could improve students' understanding of mathematics. We seek to help teachers (and researchers) identify most of the mistakes made by students in order to recognize them in their classrooms. The results indicated that some of the students' most common errors are in line with what the literature points out, that is, difficulty in translating the problem into mathematical language; difficulties in using the operations algorithms and difficulties in understanding what the problem is asking.

Palavras-chave: Erros matemáticos. Representações matemáticas. Análise de erros. Educação matemática. Anos iniciais.

Resumo: Neste artigo analisamos os erros matemáticos dos alunos dos primeiros anos de ensino na escola brasileira. Para atingir este objetivo, desenvolvemos um questionário com oito problemas matemáticos que incluíam operações básicas. Esse instrumento foi distribuído a 76 alunos da terceira série de três turmas diferentes de três escolas diferentes. Para entender o que causou os erros, analisamos cada questão. Para realizar essas análises, usamos as seguintes categorias: Erros devido a dificuldades de linguagem; Erros devido a dificuldades na obtenção de informações espaciais; Erros devido a associações incorretas ou rigidez de pensamento; Erros devido à aplicação de regras ou estratégias irrelevantes; Erros devido ao baixo domínio de habilidades, fatos e conceitos pré-requisitos; Reprodução incorreta da tarefa; Erro de contagem; Erros na montagem de operações aritméticas; Erros no resumo com valores maiores que dez; e erros de subtração específicos. Acreditamos que usar os erros como motivação pode melhorar a compreensão da matemática pelos alunos. Procuramos ajudar professores (e pesquisadores) a identificar a maioria dos erros cometidos pelos alunos para reconhecê-los em suas aulas. Os resultados indicaram que alguns dos erros mais comuns dos alunos dialogam com o que a literatura aponta, isto é, dificuldade em traduzir o problema para a linguagem matemática; dificuldades em usar os algoritmos das operações e dificuldades em compreender o que o problema está pedindo.

The beginning

The results we present are part of an international project involving professional development of mathematics teachers in Brazil. Regarding formal education in mathematics, Brazil has shown unsatisfactory levels of performance. For example, the Program for International Student Assessment – PISA reported, "Indeed, the average scores reached by students in Brazil and Peru are lower than those reached by 90% of students in OECD countries" (OECD, 2003, p.101). Test results indicate that on average, the Brazilian 15-year student knows only how to do one-step procedures of mathematics that are very basic and do not involve interpretation or reasoning.

We believe that this is a late diagnosis of problems in mathematical literacy carried over from elementary school, which deals with the formal education of children from first to fifth grade. Thus, we believe it is important to apply and analyze observation instruments that explicate difficulties in mathematics of our children and their teachers. Moreover, in collaboration with researchers from other OECD countries, we have noted the importance of perceiving similarities and differences in procedures, content, curriculum, and teaching methods between these countries and Brazil.

From 2012-2013, Pro-Literacy - Mobilization for Education Quality was developed at the national level as a "continuing education program for teachers to improve the quality of reading/writing learning and mathematics in first grades[1] of elementary school" (BRASIL, 2014, p. 1). The program was designed by the Ministry of Education in partnership with universities that joined a call from the National Network of Continuing Education, also with

[1] The Program focused the content explored in the first to third grades of elementary school.

the support of Brazilian states and municipalities. This program was offered to all teachers in public elementary schools in a "chain training"[2], which is a large nationwide program.

From this experience, we developed an international research project, which consisted of two phases:

1) Near the end of the Pro-Literacy activities, a diagnostic instrument containing math problems was distributed to teachers, so they could solve problems, and, after that, they could apply to their elementary school students. We accumulated a vast amount of material for analysis, which aims to focus on the representations of mathematical concepts from such participants in the program.

2) In 2014 the National Pact for Literacy in Right Age (Pact) started, which was also a program of the federal government, with similar characteristics to Pro-Literacy but with greater focus on interdisciplinary connections with the study of Portuguese. From the analysis carried out in the first phase of this research, we developed a new diagnostic tool, improved from the initial results of theoretical discussions and contributions of researchers of the project, including foreign participants.

In this paper we focus on the instrument of phase one, addressing the following research questions: *What are the most common mistakes committed by students? What do the errors say about the children's mathematical knowledge?*

Error analysis

We agree with Oserand and Spychiger's (2005) definition of the concept of *error* as a process or fact that does not match a given norm. An error is often seen as something bad and negative in our lives, and it would not be different in a school environment. "Many people associate negative feelings with errors, which probably arise from the fact that errors are one of the most important criteria to assess the performance of individual actions" (RACH et al., 2013, p. 22). Error analysis arises in mathematics education in order to modify this view of the students' mistakes, trying to understand them and use them as a tool for comprehending mathematical concepts.

Borasi (1996) stated that error in schools is considered problematic because it is directly associated with evaluation. She believed that the school system pressures the students, and that school failure is a result from the mistakes made by them. She also suggested that if evaluation focuses on the process, and not only on the result, but errors could also be seen as an opportunity to learn.

> To summarize, we have suggested that errors can be used as a motivational device and as a starting point for creative mathematical explanations, involving valuable

[2] The "chain training" will be described in the session 3.

> problem solving and problem posing activities. We have also suggested that errors can foster a deeper and more complete understanding of mathematical content, as well as of the nature of mathematics itself (BORASI, 1987, p. 7).

Using errors as motivation could improve students' understanding of mathematics as a discipline and could contribute to changing their thoughts about it (BORASI, 1987). However, if error is such an important part of mathematics' learning, why do teachers most of the time not use it as a positive tool? One of the possible answers for this question is based on the difficulties to comprehend what caused the error.

In the 1970s, Radatz (1979) developed an error classification, creating five categories to try to get a better understanding why a given mistake was made:

(R1) Errors due to language difficulties: this kind of error is caused due to the difficulties to translate the problems to a math language;

(R2) Errors due to difficulties in obtaining spatial information: this kind of error emerges when the student has to discover some information from visual and/or iconic representation;

(R3) Errors due to incorrect associations or rigidity of thinking: the students develop cognitive operations and continue to use them even though the fundamental conditions of the mathematical task have changed;

(R4) Errors due to the application of irrelevant rules or strategies: this error can be defined as the incorrect use of algorithms or the application of inadequate strategies in solving a task;

(R5) Errors due to deficient mastery of prerequisite skills, facts, and concepts: this kind of error includes all deficits in the content.

Almost two decades later, Batista (1995) also developed five categories to classify arithmetic errors from elementary school students. These categories emerged after a study with pupils from second to fourth grade.

(B1) Incorrect reproduction of the task: this kind of error occurs when a student reaches a certain kind of resolution (correct or not) different from what was asked for;

(B2) Error of counting: this error includes all kind of mistakes with basic operations;

(B3) Errors in the assemblage of the arithmetic operations: this kind of error can be related to a place-value error;

(B4) Errors in the summing up with values bigger than ten: this kind of mistake is also related to the lack of comprehension of the place value of the digits in the numerical decimal system;

(B5) Specific errors of subtraction: this type of error includes every mistake related to a subtraction count.

Radatz (1979, p. 164) also stated that "it is quite difficult to make a sharp separation among the possible causes of a given error because there is such a close interaction among causes", but even then, diagnostic errors seem to provide helpful information about mathematics learning.

Despite those difficulties, error analysis is a mathematics education tool used worldwide. Smith (1940) investigated what kind of mistakes are common in working with geometry problems, constructing concepts, or proving results. He realized the major mistake was the use of visualization to assume information not given.

Guillermo (1992) classified mistakes made by students from 14-20 years old and highlighted the problems with special products, like perfect squares and difference of two squares. Studying elementary pupils, Bathelt (1999) presented a test about natural numbers, fractions, and decimals and analyzed the answers, trying to comprehend what students know and what they do not.

In this paper, our goal is to analyze the answers of a diagnostic instrument containing math problems. We focused on three different classes of third graders from three different schools. We sought to understand the common errors made by pupils and to identify the possible causes of them.

Method

This paper shares some results of an international project involving mathematical concept representations. The Pro-Literacy program was developed over 2 years and included participants from all of Brazil. Different universities were chosen to be the locus of the teacher training classes, which were constructed on a chain training model. As shown in the diagram below, certain cities were selected for the training. To prepare the teachers, multiple training classes were established in each city. The classes were led by facilitators, and each included 25 teachers from the surrounding cities. These facilitators were usually coordinators of schools and during the Program were responsible for leading his/her school, sharing material, learning, and discussing the program with her/his colleagues at school. In this way, even considering the large size of the country, it was possible to achieve training for most of the public-school teachers (Figure 1).

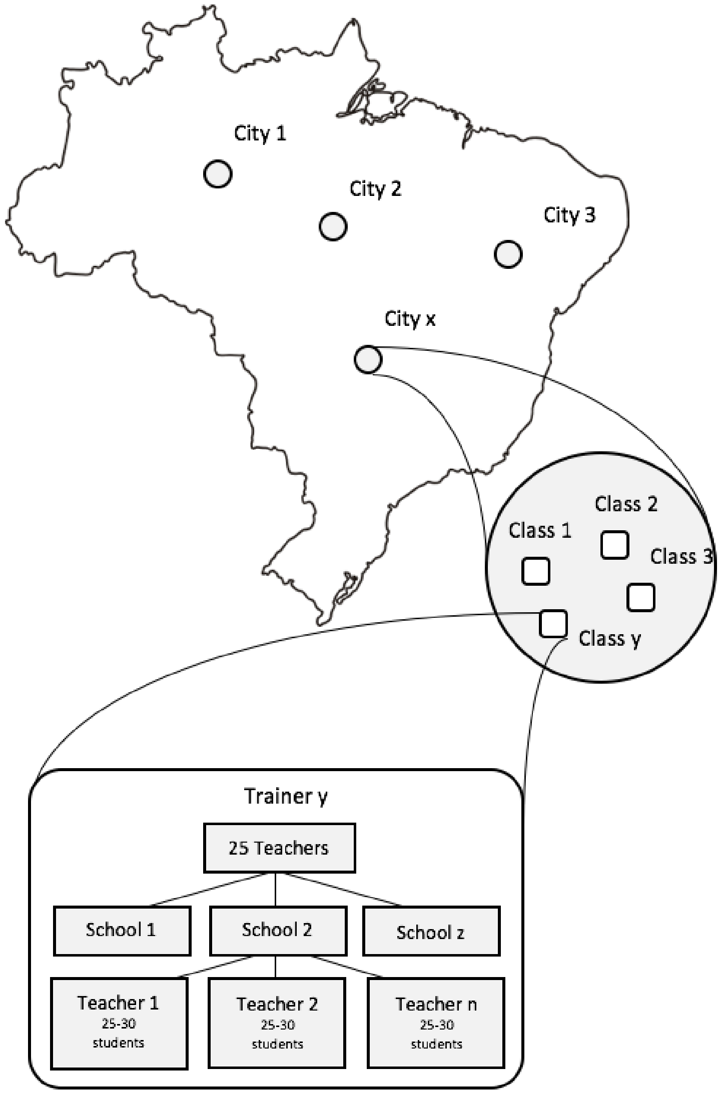


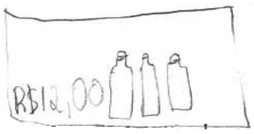
Figure 1 – The Pro-Literacy proposal

Campinas, at São Paulo State, is the city of UNICAMP - Campinas State University, and it was one of the Pro-Literacy sites. We worked here to build our set of data. In this paper, we consider the diagnostic instrument containing math problems that was distributed to three different third-grade classes from three different schools to a total of 76 students. We obtained authorization from their parents for this use. We chose the third grade because we considered it not the beginning or end of elementary school but appropriate to give a portrait of the halfway point of students' early schooling.

Results

We present the results regarding eight questions of the diagnostic instrument, one by one. Our goal is to describe the reasoning involved in each question and the kind of mistakes related to them, for the purpose of contributing to the teacher's understanding of students' thinking. A general analysis is presented at the end of the section, including the collective results.

The first question we chose to explore is "I paid $12 for three bottles of soda. How much is each bottle?" Six students did not provide an answer. Four students wrote a wrong answer (2, 3, or 12) but without anything more (no drawing or counting) to show their reasoning. Ten students wrote the right answer without counting (just number or number and drawing) like this example:



"I know the answer, but I don't know how to calculate it. Each bottle costs $4,00"

Figure 2 - Right answer without the count

Twenty-one students presented the right solution with some representation of strategy (addition, multiplication, or division) as shown in the follow examples:

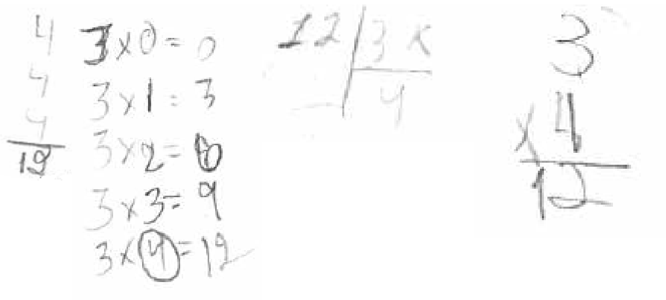


Figure 3 - Right answers with different strategies

A total of 35 students provided a wrong solution, as exemplified by Figure 3.

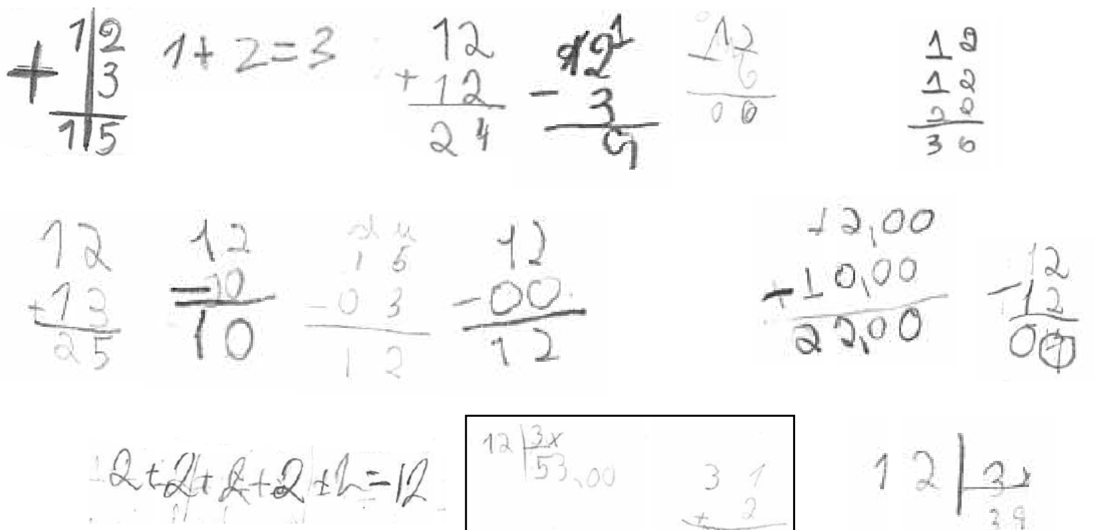


Figure 4 - Wrong answers with different solutions

The first reasoning we can note from these students' solutions is the use of the numbers of the question sentence: 12 and 3. Using them, we can consider the "15", "9", and "36" answers relating to addition, subtraction, and multiplication (or triple sum of 12). We can propose the reasoning of 11 as an answer to "12-3" where in the middle of the process the student in fact solved "13-2", switching the "2" and "3" order (probably because 3 is larger than 2).

We also want to explore the critical view of students. On this problem, it is especially relevant because it is a situation of our daily life, a contextual problem. If a person pays $12 for three bottles of soda, how can each of them be $15 or $36? We can see the students are giving the answer without considering whether it makes sense. No critical position is presented on it. We could suppose it is a mechanical work on task, and this result certainly teaches it is an important point to be attacked in the classes.

As suggested by Batista (1995) and Cury (2007), it is necessary to use tests to try to understand our students' reasoning. We highlight the importance of hearing what pupils have to say because considering a task just right or wrong is insufficient to forward the process of teaching and learning. For some solutions we can guess students' intent, as in the 9, 15, and 36 answers discussed above, but we could not find an explanation for answers like "12-12=0", "12-2=10", "7-4=4", or "12+10=22". Without seeking to embarrass students, it is important to make them talk about their solutions to try to understand their reasoning, and later help them to change it to a correct way.

> Russian psychologist Krutetskii (apud CURY, 2008) shows in his work the importance of analysing the process and not just the product. As exemplified by the author, one should not only evaluate the alternative indicated in a multiple-choice question or the result presented in an open question; It is also necessary to analyze the reasoning presented during the process of resolving the issue. Analysing the process in this way, it is possible to understand the students' mathematical abilities, in addition to the difficulties they present. The researcher also states that, in this form of analysis, students can be questioned about mistakes made and help them reconstruct knowledge (Ramos, 2015, p. 135).

Usually knowing the steps of the student thought process makes it possible to identify "what is the problem/detour" of the solution way. Graphic 1 summarizes the Question 1 answers.

Graphic 1 - Answers of Question 1
Source: Made by authors

Question 2 proposes the problem "Mary bought a doll for $4.00 and in her wallet left $7. How much did she have before the purchase?" Two students did not answer this question. Five students presented the right answer without any counting or drawing, which makes it impossible to discern their reasoning.

Fifty-one students completed a correct solution, presenting two kinds of reasoning - a sum of 7 and 4 (7+4=11) or a subtraction of 4 from 11 (11-4=7). That was probably solved mentally first to find 11 as the answer and then present the counting that represents the situation - has $11 in the wallet initially, spends $4 buying the doll (subtraction idea), leaving $7 in the wallet at the end.

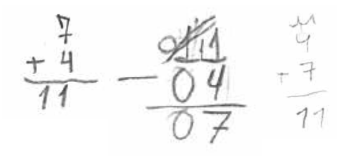


Figure 5 - Right answer with sum or subtraction strategies

On this contextual situation, Mary is spending money, and to calculate the answer it is necessary to sum the numbers of the sentence, which can be confusing for some students. The usual idea we have about spending money is to use subtraction, because we have less money after the purchase. This was the supposed reasoning of 15 students who presented a wrong solution based on "7-4" or "4-7" (note this last was the order of the number of the sentence). One student multiplied these numbers, and two found a wrong sum.

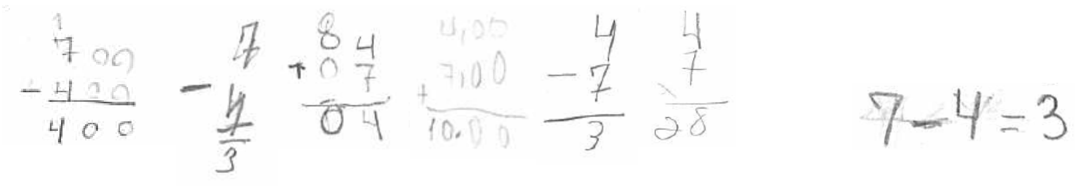
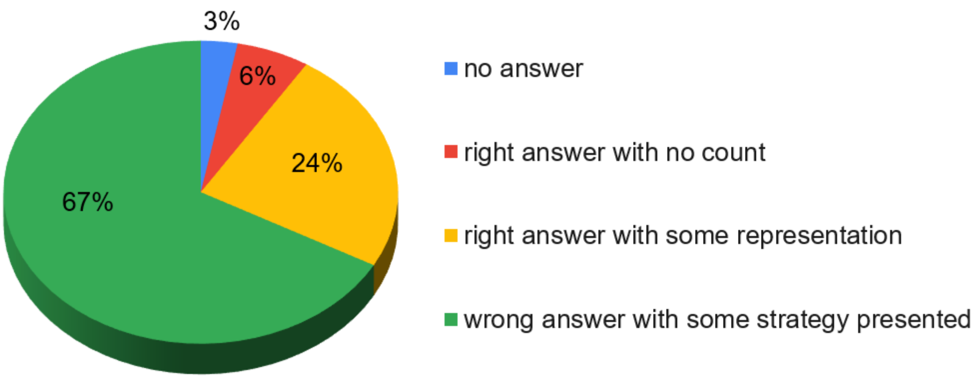


Figure 6 - Wrong answers and different solutions



Graphic 2 - Answers of Question 2
Source: Made by authors

Question 3 presents the follow situation: "Juca has $10 and wants to buy packs of candy per $2 each one. How many packs can he buy with this money?" Six students did not answer anything. Fourteen students provided the right answer writing just the number or making a drawing like this below (Figure 7).

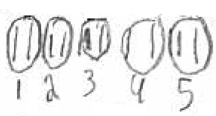


Figure 7 - Drawing of representation of solution

Thirteen students calculated the correct answer using division, addition, multiplication, or subtraction.

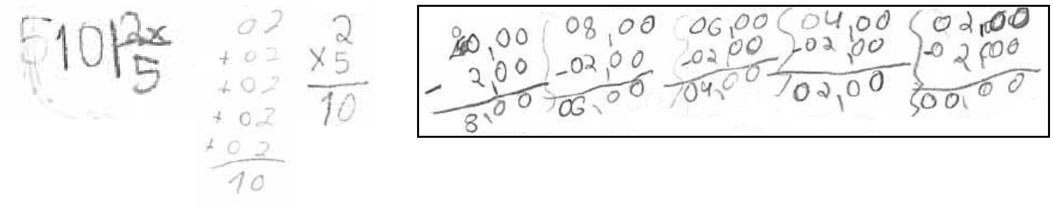


Figure 8 - Right answers with different strategies

Almost half of the students (36) showed some counting but did not calculate the right answer. As we discussed before, most of them tried to count using the numbers of the sentence: 10 and 2 ("10-2", "10+2", or "10x2"). We note some of them used a vertical counting method, putting number 2 in the "ten" spot, which is the reason we suppose they had 30 as an answer. This point is relevant to identify the difficulty with positional value by some students.

Again, we can highlight the importance to explore the students' critical view during the classes. Here we could focus on two points: a) If Juca had $10 and each pack of candy is $2, how it is possible to buy 20, 22, 30, or 50 packs? b) Some students answer how much Juca spent/saved (for example: "he spent $20", "he saved $17", "it costs $30"). They did not recognize what the question was asking.

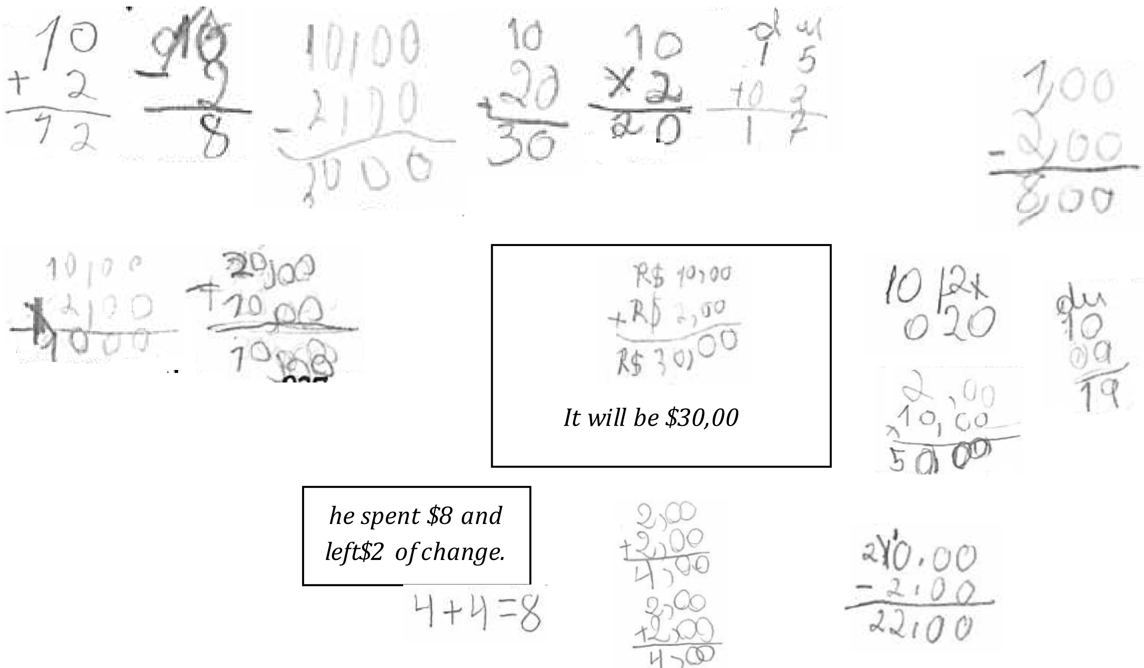
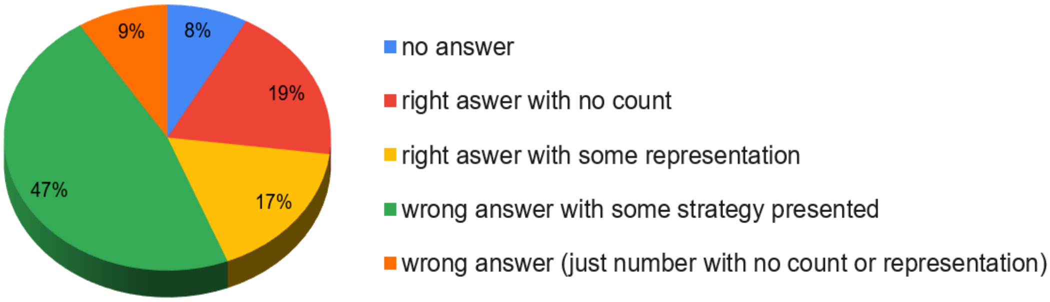


Figure 9 - Wrong answers and different representations of reasoning



Graphic 3 - Answers of Question 3
Source: Made by authors

The fourth question we want to explore is "I have 3 packs of yogurt. Each pack includes 4 yogurts. How many yogurts do I have?" One student did not try to answer. Fifteen students presented the right answer based on some representation or just showing the number.

Figure 10 - Right answer with some representation

Six students wrote incorrect answers without a solution, which makes it difficult to understand how they got the numbers 1, 3, 11, 14, and 18 as answers. Again, we highlight the importance of talking to the students about their solutions and not just noting "right" or "wrong" on their tasks or tests.

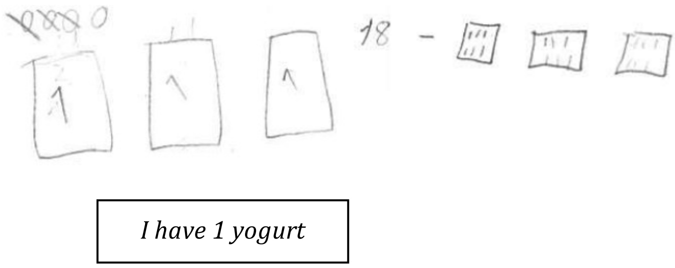


Figure 11 - Wrong answer with some representation

A total of 28 students found the right answer based on one of these two solutions:

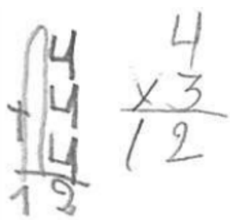


Figure 12 - Right answer

Twenty-six students did not calculate the right answer. As we discussed previously, most of them tried to use the numbers from the sentence of the question. What is possible to identify in this question is errors related to their calculation process: We can see different results for "3 x 4" (13, 16, and 19). Multiplication is a new operation for this grade so probably they are still confused.

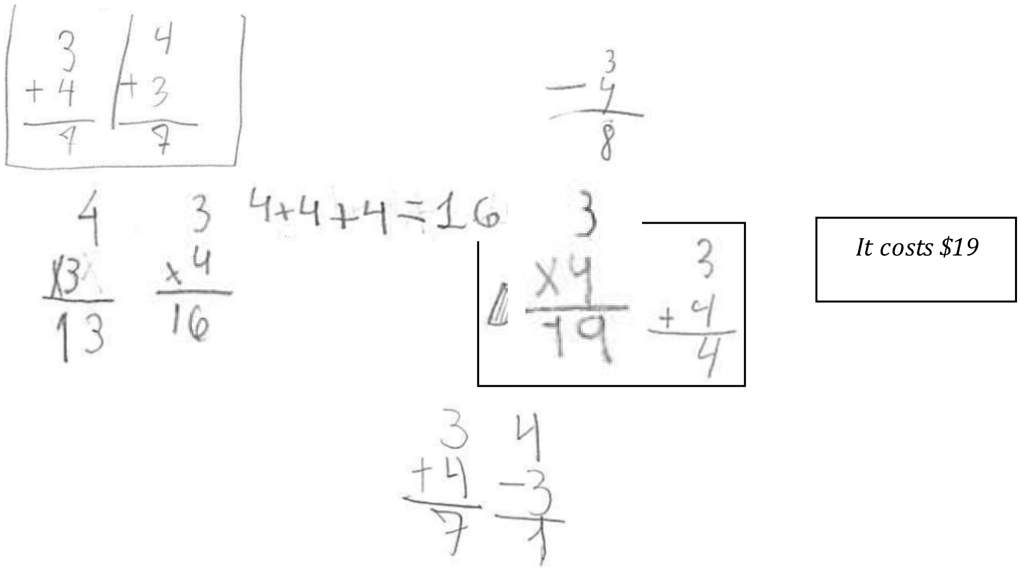
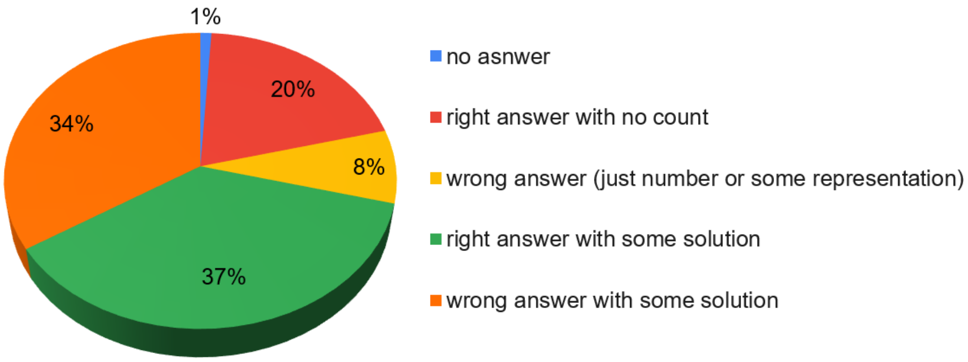


Figure 13 - Wrong answers with different strategies



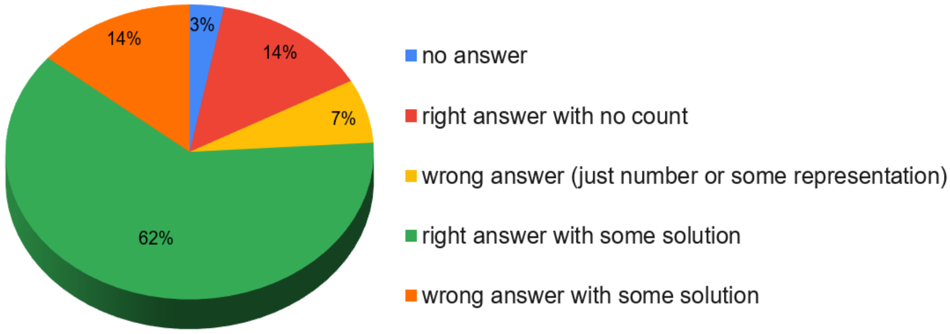
Graphic 4 - Answers of Question 4
Source: Maed by authors

Question 5 proposed: "Ana is 8 years old and Carlos is 2 years older than her. How old is Carlos?" Two students did not answer the question. Eleven wrote the number of the answer but without any representation. Five students answered incorrectly without solution, so that it is not easy to identify their reasoning (for example: "He is 2 years old" or number 9, 11, and 12). For this problem 47 students calculated the right answer by counting "8 + 2". Another 11 students presented wrong answers with different solutions:

Figure 14 - Wrong answers with different solutions

We can see the same situation discussed earlier, confirming our supposition that most students try to use the numbers of the sentence to solve the problem, even without knowing exactly what to do (which operation).

Two students wrote a sentence that it is not the answer: "He is 6 years younger" and "It costs $16", which again highlights the importance of exploring with students the critical view around the problem situation.



Graphic 5 - Answers of Question 5
Source: Made by authors

The sixth question is "Mary had some cookies and received 4 cookies more from her grandma, having a total of 12. How many cookies did Mary have in the beginning?" Four students did not answer anything. Ten students wrote the right answer but without counting (just number or number and drawing) like the example in Figure 13:

She had 8 cookies

Figure 15 - Right answer without solution

Twelve students wrote a wrong answer (4, 5, 6, 9, 10) but without anything more (no drawing or counting shown). One pupil wrote, "She has 6 cookies, but I do not know how to calculate it." As we said before, in this kind of situation it is hard to guess his reasoning.

Twenty-three students presented the right answer with one of these solutions:

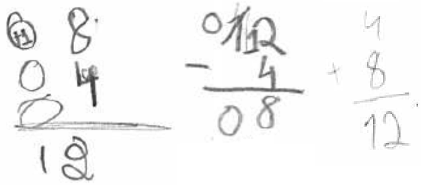


Figure 16 - Right answers with different solutions

For this question 30 students calculated incorrect answers, presenting different strategies. Once more, we see that sentence numbers (4 and 12 for this question) are included in most of counts. As Ana "received more cookies", the natural big idea is the sum, so 15 students chose 16 as an answer. As happened previously, we see multiplication, addition, and subtraction with sentence numbers, and we can suppose students do not know exactly what to do with them, just know they should use them. The difficulty with the place value also appears again (look at "4+12=52" as an example).

As mentioned previously, we could explore with the students the critical view: If Mary had 12 in the total, after receiving 4 cookies from grandma, how could she have had 32 or 16 in the beginning, for example? They are not reflecting about the reasonableness of their answer relative to the problem situation. An answer "it costs..." shows no focus on the question.

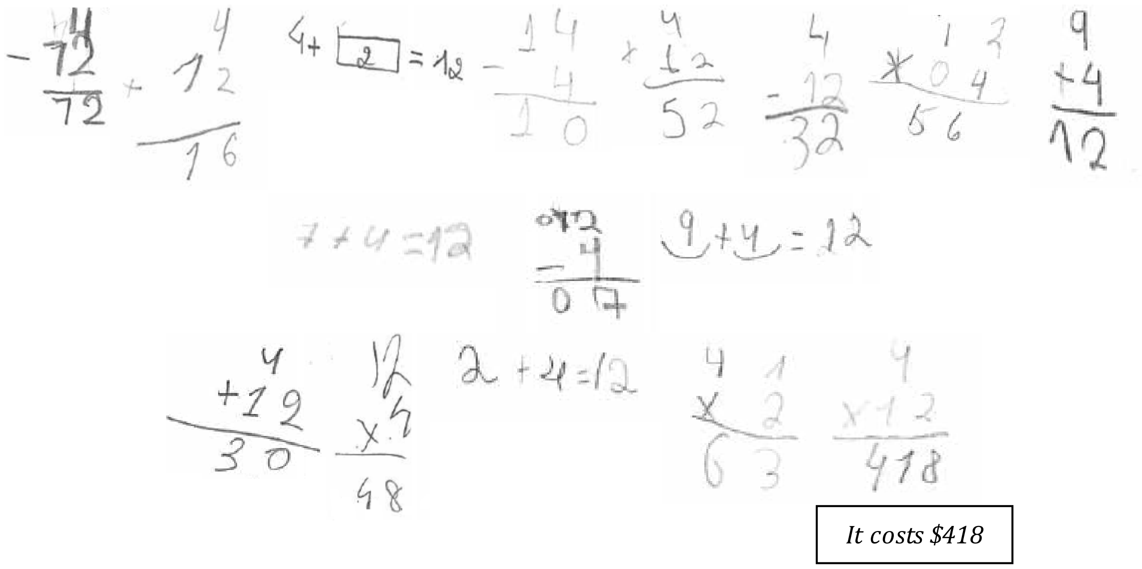
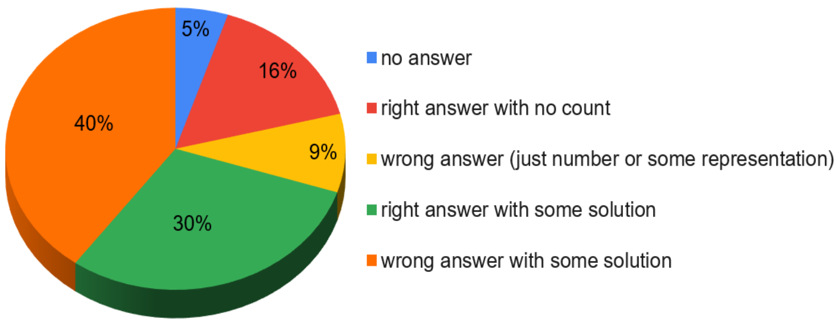


Figure 17 - Incorrect answers and different solutions



Graphic 6 - Answers of Question 6
Source: Made by authors

The seventh question is "The owner of a bookstore had 937 books in stock. He got some more and now he has 1254 books. How many books has he acquired?" Two students did not try to answer. One student wrote the right answer but without solution. Three students wrote wrong answers (for example "one hundred thousand") without showing any calculations to reveal their reasoning. Sixteen students presented the right solution based on the same count (subtraction): 1254 - 937 = 317.

For this question 54 students calculated a wrong answer, presenting 28 different solutions.

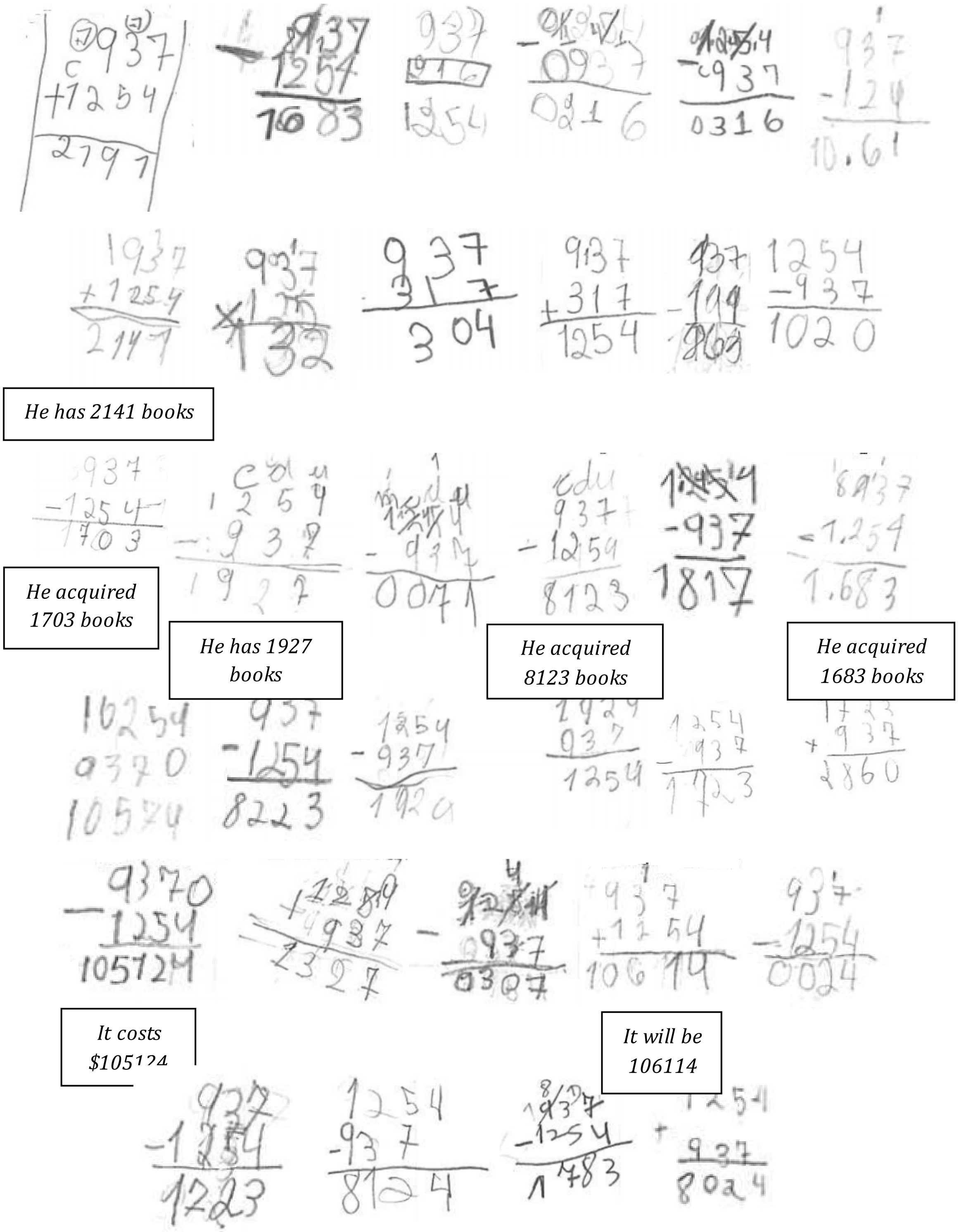


Figure 18 - Wrong answers with different solutions

We can see in this question how complicated counting can be for a child. The instrument we prepared was applied to classes from third to fifth grade. This count is difficult for a student of third grade because of the numbers used. It is possible to identify here some points we have already mentioned, such as the use of numbers 1254 and 937 (presented in the sentence of the question) even though they do not really know what to do with them (which operation). With three- and four-digit numbers the problem with place value is more evident. Cases like this go in the direction of what Mendes (2007, p. 55) calls the structure of number.

> The number structure category concerns the error due to a lack of understanding of the number structure, grouping the positioning that it generates in the operating procedures. The origin of the error is not in the operation itself, but in the structure of the number understood by the student. It may be the result of misunderstanding or meaning of the decimal structure, which may result in errors in operating procedures.

For example, we can look at 1254-937=8124.

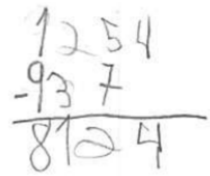


Figure 19 - Example of mistake involving place value

Number 9 was incorrectly placed in the far-left column (thousands position). As happened previously, we can recognize as the student found it impossible to solve "1-9", they just did the opposite "9-1". The same applies to "3-2" and "7-5". However, in the "ones" position, number 4 is larger (than number 0 for the empty spot of ones in this representation of 937), so now it is "4-0" independently of the order (what is important is taking larger minus smaller). This rule "bigger minus smaller" is presented in more than one solution, as shown below.

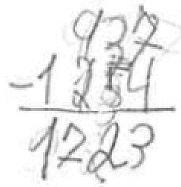


Figure 20 - mistake using idea of "bigger minus smaller"

In this example the student always took "bigger minus smaller" too, taking "7-4=3" at the ones position, "5-3" on tens, "9-2" on the hundreds, and finally "1-0" for the thousands. Sometimes the student can recognize some rules of the subtraction operation but it is not fully clear, probably because the student did not understand the process, just tried to follow the rule.

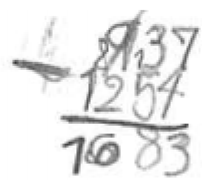


Figure 21 - second example of mistake using idea of "bigger minus smaller"

In this example above, the child can identify that the rule is "minuend minus subtrahend", starting with "7-4", "3-5" that is necessary to change to "13-5", which gives 9 that was miscounted as 8, but in the end, there is nothing in the minuend's thousands, so the student considered the opposite order, "1-0". He/she failed to recognize the order of the numbers necessary to have the correct position vertically (that should be 1254 - 937).

One more time we highlight the necessity of hearing from the students, not just analyzing what they produced on the assessment. Considering these examples of Figures 17, 18, and 19, it is possible to suppose their reasoning, and work on it to try to help the student to not repeat it. However, as we mentioned earlier, some mistakes are difficult to understand, such as Figure 22. How do the ones and hundred positions become zeros in the answer?

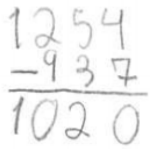


Figure 22 - Example of a difficult situation to understand the mistake

The critical view is explicitly an opportunity in this situation to be explored in the classes. Again, we have numbers that do not make sense to the problem (if in the total owner had 1254, how could he have acquired 1683, 1927, 2191, 8223, or 105124 books?).

The graphic below summarizes the solutions of Question 7:

Graphic 7 - Answers of Question 7
Source: Made by authors

The last question to analyze is "Eight times which number is two?" No student found the right solution for this question. A total of 35 students did not answer this question or wrote notes like "eight times twelve is the right question", "I thought it is very very hard", "no number takes two", or "I didn't understand". The other 39 students produced some form of incorrect solution. Figure 21 shows some examples that prominently feature the sentence numbers 8 and 2. It is possible to suppose they thought about 16 | 8 = 2, considering the division process is the inverse process of multiplication. On the other hand, we can imagine they tried to use both numbers in some incorrect way (e.g.,8 - 6 = 2). Neither is a right answer for the question, but at least these students can identify some information from the question that could have led them to a correct count.

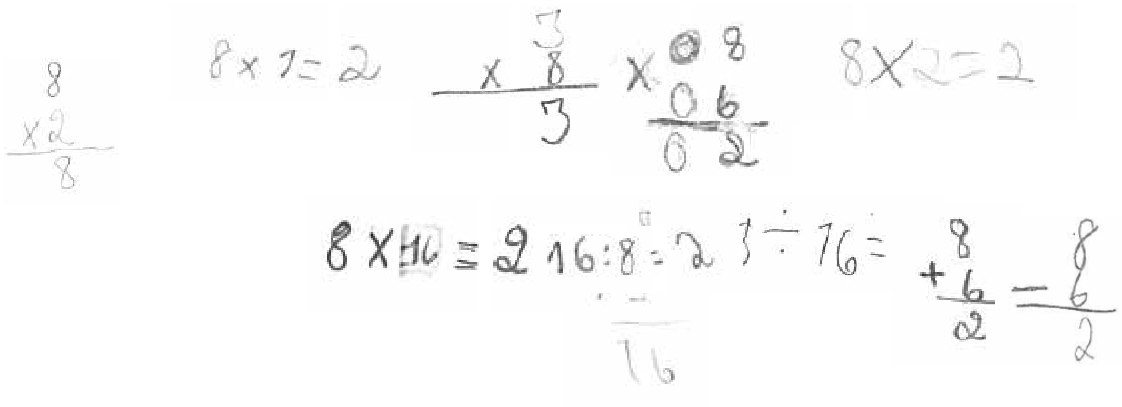


Figure 23 - Answer using numbers 8 and 2

The student whose work is shown in Figure 23 suggested that we should correct the question. He went further than other students, not just giving a way to organize the numbers but suggesting one way to make it right: correcting the question.

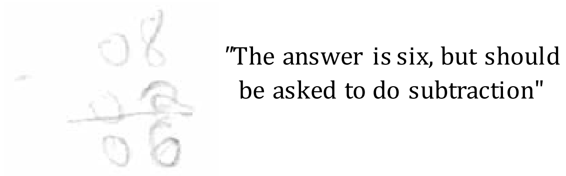
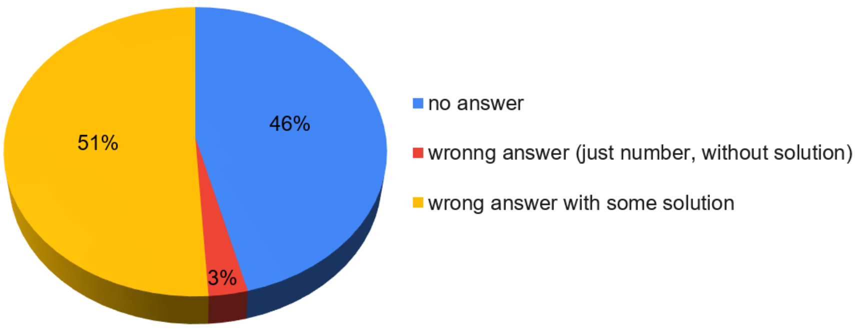


Figure 24 - student suggesting to change the question

Two students just answered with a number ("4" and "10"). To build the graphic of Question 7 we grouped them with the other students who tried to solve the problem but were not correct:



Graphic 8 - Answers of Question 8
Source: Made by authors

Having presented a discussion of each question of our research instrument to identify and recognize the principal mistakes made by students, we now turn in the next section to an analysis involving the collective results.

Discussion

To discuss the data presented previously, we decided to use categories developed by two different researchers, Radatz (1979) and Batista (1995). Both authors considered it important to look to student errors as a tool to enhance the learning of mathematics and, to do so, they discussed the types of mistakes students usually commit.

On the first question, we can find several types of errors like the ones described above. For example, if we look at Figure 25 we can identify the categories R1 and B2. The student who did the first count (Figure 25a) probably had difficulty in translating the problem to a math language, so he added the number 12 three times instead of trying to understand how many times the number 3 fits in the number 12. This situation is similar to what Lopes and Kato (2011, p. 5) identify in their research. The authors say they consider "that certain obstacles that arise during problem solving are linked to the decoding of specific mathematical terms that appear in their statements". It is important for teachers to understand whether the difficulty lies in interpreting or solving a certain algorithm, for example.

The pupil who did the second count (Figure 25b) probably has difficulty comprehending the concept of division (and multiplication). He/she understood that to solve the problem it would be necessary to divide 12 by 3, but he did not know how to calculate it.

(a)

Figure 25 – Types of errors of the first question

On the second question, we can find the categories R1, R3, B2, and B5. The student who did the resolution (a) (Figure 26a) was not able to translate the problem to a math language, so instead of adding the numbers 7 and 4, he performed a subtraction. This is a problem that involves buying something, or in other words, spending money. It is common in this kind of exercise to think that they have to do a minus count to find how much money is left, but that was not the case with this question. The student had an idea about a certain type of problem and transferred it to another problem, even though it did not have the same fundamental bases (R3- rigidity thinking). The second resolution (Figure 26b) shows a pupil's error related to counting. The student committed an error of counting in the hundreds place.

(a) **(b)**

Figure 26 – Types of errors of the second question

On the third question, we found the errors located on categories B1, B4, and B5. One error, in particular, is very appealing (Figure 27). The pupil decided to subtract two from ten and, to do so, he/she used the traditional subtraction method. In this method, the columns are processed from right to left, and since the subtrahend digit (2) is greater than the minuend digit (0), the student had to borrow ten from the column on the left. Instead of borrowing, he/she added ten, causing the solution below.

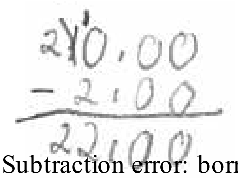
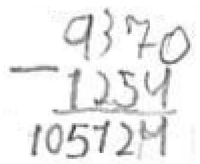


Figure 27 – Subtraction error: borrowing ten

Young and O'Shea (1981) have reported that this kind of error is common among children. Students usually use the "borrow technique" even if they do not need it. Probably this mistake is related to a misunderstanding of the concept of subtraction. An error analyzes can help the teacher to comprehend what to do to help the students who have this kind of doubt.

The fourth, fifth, sixth, and seventh questions have the same kind of mistakes. Some students did not answer what was asked for in the question. The seventh question, for example, asked, "The owner of a bookstore had 937 books in stock. He got more and now he has 1254 books. How many books has he acquired?", and one answer was the phrase "it costs $ …" (see Figure 28).



It costs $105124

Figure 28 – Answer that does not relate to the question

This type of error leads us to think about the importance of discussing with students the meaning of their answers and the necessity to reflect on the response obtained in a given problem. The student should note that particular response is not valid in a given problem. They have to check whether that answer makes sense in that particular context. It a good opportunity to use mathematics to improve the critical view during classes.

Dialogue has to be an important function in the class to explore the critical view and to understand our students better. In this paper we have sought to contribute to teachers (and researchers) identifying most of the mistakes made by the students, in order to recognize them and then try to teach in a way to clarify misunderstandings, so that students do not repeat the same mistakes. But as we discussed during the presentation of the solutions, sometimes it is impossible to categorize the reasoning, particularly when students record answers without showing their solution methods. These examples show us the necessity of hearing our students explain their reasoning, which will in turn help us in recognizing mistakes and clarifying the doubts.

To finish, we highlight that to employ these processes appropriately it is necessary to emphasize the teacher's role. Teacher training (both preservice and in-service) has to explore the importance of the analysis of student errors, understanding how they can be a support to organize the class. If teachers are concerned about the types of mistakes made by students, they can address the problematic aspects with their students, focusing on the errors and clarifying them.

Conclusions

It is important to highlight our goal is to deeply understand what causes student errors, showing and discussing the possible reasoning for them. The distribution of errors shown in the graphics helps to illuminate which kind of questions present students more difficulty. We do not want to emphasize the mistakes by themselves, but support teachers to identify which content creates most of the difficulties for students and therefore should be better explored in the classes.

Using previously identified classifications of errors, we analyzed errors of students who completed our diagnostic instrument support teachers to work on the important points that arose. Considering it is impossible to identify all the mistakes, particularly when students do not show their reasoning in their work, dialogue can be important in the class, for students to share their reasoning so that class discussions can help prevent future errors.

We present these reflections with the intention to contribute to teachers' (and researchers') thinking about the important role of student errors, how they can be incorporated into teacher training, and the importance of classroom dialogue for the educational process.

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