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IS S5 PARACONSISTENT?

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Abstract. The aim of this note is to examine the claim of Béziau that **S5** is paraconsistent. In particular, I wish to clarify what exactly is implicitly assumed behind his claim, and how we might be able to assess the claim "**S5** is paraconsistent".

Keywords: negation • paraconsistent logic • paracomplete logic • modal logic

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1. Introduction

In (Béziau 2002), Jean-Yves Béziau claimed that the well-known modal logic **S5** is paraconsistent.[1] The trick behind his claim is that negation is understood as a combination of modality and negation of the form "not necessarily" or "possibly not". This idea works, since the following formula is *not* valid/provable in **S5**:

$$p \to (\Diamond\neg p \to q).$$

The aim of this note is to examine the above claim of Béziau. In particular, I wish to clarify what exactly is implicitly assumed behind his claim, and how we might be able to assess the claim "**S5** is paraconsistent".[2]

For preliminaries, our languages $\mathscr{L}$ and $\mathscr{L}_m$ consist of sets $\{\sim, \wedge, \vee, \to\}$ and $\{\neg, \wedge, \vee, \to, \Box, \Diamond\}$ of propositional connectives, respectively, and a countable set Prop of propositional parameters. Furthermore, we denote by Form and Form_m the sets of formulas defined as usual in $\mathscr{L}$ and $\mathscr{L}_m$ respectively. We denote formulas of both languages by A, B, C, etc. and sets of formulas of both languages by Γ, Δ, Σ, etc. Finally, we assume logics to be closed under *Modus Ponens* and uniform substitution of arbitrary formulas for propositional variables.

2. The main worry

The main worry is based on a simple observation that is also made by Béziau (2002, p.5). That is, given the modal vocabulary, "possibly not" is not the only negative modality. Indeed, the more popular negative modality is "necessarily not".[3] What we have in this case is a reason to conclude that **S5** is paracomplete, but not paraconsistent, given the following facts:

$$\vdash_{\mathbf{S5}} A \to (\Box\neg A \to B),$$
$$\nvdash_{\mathbf{S5}} p \vee \Box\neg p.$$

Therefore, in order to defend the claim that **S5** is paraconsistent, one needs to argue in favour of "possibly not" over "necessarily not". On which ground this can or cannot be done is a question that seems to be related to the issue of interpreting paraconsistency. However, I will leave the question as a topic for interested readers. Still, I would like to add three observations that are motivated by the worry here, and it might also be of some help to address the question.

3. Observation (I)

The worry raised above can be seen as a tension between the behaviour of "possibly not" and "necessarily not" in light of paraconsistency. Note that modal logics are not *always* paraconsistent by looking at "possibly not". Indeed, we obtain the following.

Proposition 1. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$.[4] *Then, for all* $A, B \in \mathsf{Form}_m$,

$$\vdash_{\mathbf{L}} A \to (\Diamond\neg A \to B) \text{ iff } \vdash_{\mathbf{L}} \Diamond\neg A \to \neg A.$$

Proof. The right-to-left direction is immediate. For the other direction, note that $\vdash_{\mathbf{L}} A \to (\Diamond\neg A \to B)$ iff $\vdash_{\mathbf{L}} \Diamond\neg A \to (A \to B)$, and if we instantiate B by $\neg A$, then we obtain $\vdash_{\mathbf{L}} \Diamond\neg A \to (A \to \neg A)$. Since we have $\vdash_{\mathbf{L}} (A \to \neg A) \to \neg A$, we obtain the desired result. □

Remark 2. *Note that the above proof does not require full classical logic, but intuitionistic logic, for example, will suffice.*

Similarly, for "necessarily not", we obtain the following result.

Proposition 3. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$. *Then, for all* $A, B \in \mathsf{Form}_m$,

$$\vdash_{\mathbf{L}} A \to (\Box\neg A \to B) \text{ iff } \vdash_{\mathbf{L}} \Box\neg A \to \neg A.$$

Proof. The right-to-left direction is obvious. The other direction is similar to the proof of the previous proposition. □

Before combining the above results, let us introduce the following notion.

Definition 4. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$*. Then,* **L** *is* paraconsistent *iff* $\nvdash_{\mathbf{L}} p \to (\Diamond\neg p \to q)$ *and* $\nvdash_{\mathbf{L}} p \to (\Box\neg p \to q)$*.*

Remark 5. *This definition reflects the worry raised against Béziau's clam that "***S5** *is paraconsistent". Indeed, according to the above definition,* **S5** *is* not *paraconsistent since the second condition is not satisfied.*

Then, we can summarise the first observation as follows.

Proposition 6. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$*. Then,* **L** *is paraconsistent iff* $\nvdash_{\mathbf{L}} \Diamond\neg p \to \neg p$ *and* $\nvdash_{\mathbf{L}} \Box\neg p \to \neg p$*.*

If we assume two more conditions, then we can simplify further.

Corollary 7. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$ *such that* $\vdash_{\mathbf{L}} \Box A \leftrightarrow \neg\Diamond\neg A$ *and* $\vdash_{\mathbf{L}} \Diamond A \leftrightarrow \neg\Box\neg A$*. Then,* **L** *is paraconsistent iff* $\nvdash_{\mathbf{L}} p \to \Box p$ *and* $\nvdash_{\mathbf{L}} p \to \Diamond p$*.*

Proof. For the first condition, this is immediate in view of the following equivalence: $\vdash_{\mathbf{L}} \Diamond\neg p \to \neg p$ iff $\vdash_{\mathbf{L}} p \to \neg\Diamond\neg p$ iff $\vdash_{\mathbf{L}} p \to \Box p$. For the second condition, the following equivalence will suffice: $\vdash_{\mathbf{L}} \Box\neg p \to \neg p$ iff $\vdash_{\mathbf{L}} p \to \neg\Box\neg p$ iff $\vdash_{\mathbf{L}} p \to \Diamond p$. □

Note also that when **L** is paraconsistent with the two additional conditions, it turns out that it is also paracomplete, in the following sense.

Definition 8. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$*. Then,* **L** *is* paracomplete *iff* $\nvdash_{\mathbf{L}} p \lor \Diamond\neg p$ *and* $\nvdash_{\mathbf{L}} p \lor \Box\neg p$*.*

Then, we obtain the following result.

Proposition 9. *Let* **L** *be an expansion of classical logic in* $\mathscr{L}_m$ *such that* $\vdash_{\mathbf{L}} \Box A \leftrightarrow \neg\Diamond\neg A$ *and* $\vdash_{\mathbf{L}} \Diamond A \leftrightarrow \neg\Box\neg A$*. Then,* **L** *is paraconsistent only if* **L** *is paracomplete.*

Proof. Note first that with the two additional conditions, we obtain both $\vdash_{\mathbf{L}} \Box\neg\neg A \leftrightarrow \Box A$ and $\vdash_{\mathbf{L}} \Diamond\neg\neg A \leftrightarrow \Diamond A$. Suppose **L** is paraconsistent. Then, in view of Corollary 7, we also obtain $\nvdash_{\mathbf{L}} \neg p \to \Box\neg p$ and $\nvdash_{\mathbf{L}} \neg p \to \Diamond\neg p$. Indeed, assume for reductio that $\vdash_{\mathbf{L}} \neg p \to \Box\neg p$. Then, by uniform substitution, we obtain $\vdash_{\mathbf{L}} \neg\neg p \to \Box\neg\neg p$, and therefore $\vdash_{\mathbf{L}} p \to \Box p$, but this contradicts the assumption. The other condition can be proved in a similar manner. □

In view of these results, we may conclude that there is a wide variety of modal logics that can be regarded as paraconsistent, with a strengthened condition compared to the one considered by Béziau, and they are also paracomplete.[5] But are there particularly interesting classes of modal logic that are paraconsistent and paracomplete? The answer seems to be yes, and I will turn to explain this in the second observation.

4. Observation (II)

The basic system I would like to pay attention to is an extension of the modal logic **K**, known as $\mathbf{KDD}_c$.[6] This system can be motivated as follows. First, assume the modal logic **K** for the sake of simplicity. Second, our main worry concerned the tension between the two negative modalities "possibly not" and "necessarily not". But this tension can be resolved in a rather simple manner if we assume the equivalence of "possibility" and "necessity". As a result, we obtain the following.

- $\nvdash_{\mathbf{KDD_c}} p \to (\Box\neg p \to q)$
- $\nvdash_{\mathbf{KDD_c}} p \vee \Box\neg p$
- $\vdash_{\mathbf{KDD_c}} \Box\neg(A \wedge B) \leftrightarrow (\Box\neg A \vee \Box\neg B)$
- $\vdash_{\mathbf{KDD_c}} \Box\neg(A \vee B) \leftrightarrow (\Box\neg A \wedge \Box\neg B)$

Note further that there are plenty of systems that extend $\mathbf{KDD}_c$. This can be easily seen from the perspective of the model-theoretic semantics. Indeed, somewhat informally speaking, one can let it hop for m steps, and start to loop with the period n, for $m, n \geq 0$, for the functional modality. More precisely, if we let $\langle W, R\rangle$ be the Kripke frame for $\mathbf{KDD}_c$, then there are systems that are obtained by adding the following frame condition, for $m, n \geq 0$.

$$R^m(w) = R^{m+n}(w) \text{ for all } w \in W$$

The differences will be reflected on the behaviour of iterated negative modality. For the details of the extensions of $\mathbf{KDD}_c$ in view of the model-theoretic semantics, see Segerberg (1986).[7]

Against the background of the first observation, I here pointed out that there is an extension of the modal logic **K** that can be seen as an instance of modal logic that can be regarded as paraconsistent. Still, the system is not completely free of concerns, and I will turn to that point in the next observation.

5. Observation (III)

So far, our attention has been paid only to the negative modalities such as "possibly not" and "necessarily not". However, for most modal logics, there are infinitely many varieties of iterated modalities. For example, if we consider negative modalities in $\mathbf{KDD}_c$, we can think of negative modality $\Box^n\neg A$ for every $n \geq 1$. Given that all of these negative modalities can also give us a reason to conclude that $\mathbf{KDD}_c$ is paraconsistent (and paracomplete), and that these negative modalities are not necessarily equivalent, we may face another worry: which negative modality should we pick as *the* negation?[8]

This worry may give rise to consider the system in which there are *no* varieties of iterated modalities. One of the ways to answer to this question and to implement the requirement is to consider an extension of $\mathbf{KDD}_c$ that is obtained by adding $A \to \Box\Diamond A$, namely $\mathbf{KDD}_c\mathbf{B}$. Then, in addition to the results listed above, we also obtain the following.[9]

- $\vdash_{\mathbf{KDD}_c\mathbf{B}} \Box\neg\Box\neg A \leftrightarrow A$

Therefore, $\mathbf{KDD}_c\mathbf{B}$ is both paraconsistent and paracomplete, and moreover, the double negation laws as well as the de Morgan laws are fully satisfied. This will be a (much belated) response to a question raised by Béziau who wrote:

> It would be also interesting to consider what kind of modal logics are associated, according to our definition, to De Morgan's paraconsistent logics. (Béziau 2002, p.6)

In fact, $\mathbf{KDD}_c\mathbf{B}$, is definitionally equivalent to an expansion of **FDE**, obtained by adding the classical negation, known as **FDEP** (cf. Zaitsev 2012), **BD+** (cf. De and Omori 2015, Kamide and Omori 2017), or more recently, as **SE4** in Avron (2020). The latter systems that expand **FDE** are presented in terms of the four-valued semantics in the above papers, but the connection becomes more clear by considering the star semantics for **FDE**, expanded by boolean (or classical) negation.[10]

In view of these, we can see that $\mathbf{KDD}_c$ is an interesting system serving as the basis for **FDE**, or Belnap-Dunn logic, especially as a generalization of the star semantics.[11]

6. Concluding remarks

This short note was motivated by the claim by Béziau that **S5** is paraconsistent. I casted a doubt if this is the case without qualification since we can not only consider "possibly not", but also "necessarily not". If one has an argument to defend "possibly not" over "necessarily not", then Béziau's claim will stand. However, without such kind of argument, Béziau's claim seems to be not without problems.

We then observed that there are a number of modal logics that will avoid the problem of Béziau's claim. Indeed, for extensions of modal logic **K**, for example, all systems without axioms T and $\mathrm{T_c}$ will be qualified as paraconsistent (and also paracomplete).[12] Among such systems are those with functional modalities, and these can be seen as a generalization of **FDE** with the star semantics.

Needless to say, this note is far from being conclusive, and that is the case since it is meant to be an invitation to further investigations. At this point, I can think of at least three directions.

First, our understanding of modal logic is rather weak.[13] The second and the third observations assumed the modal logic **K** as the basis, but this is just a choice I made for the purpose of brevity. If there are particularly interesting results to be observed for other family of modal logics remains to be explored.

Second, the *constructive* counterpart of the whole story might be also interesting and fruitful. Indeed, if we add the symmetric functional *negative* modality to positive intuitionistic logic, then as observed in Odintsov and Wansing (2020), we obtain a semantics for the system **HYPE**, introduced by Hannes Leitgeb in Leitgeb (2019).[14] Moreover, the basic observations in §3 starts to be more complicated since we do not necessarily have the equivalence of $\neg\Box A$ and $\Diamond\neg A$ in intuitionistic modal logic, and therefore the definition of intuitionistic modal logics being paraconsistent may come in degrees.

Finally, the definition of paraconsistency that builds on negative modalities (cf. Definition 4) may be subject to some criticisms. Indeed, in most modal logics, there are infinitely many iterated modalities, and therefore one may require more conditions for modal logic to be regarded as paraconsisitent/paracomplete logic. One such path was explored in §5 in which the iterated modalities are simplified, but this does not exclude other possibilities, and that remains to be seen.

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Notes

[1]In view of the well-known relation between **S5** and the monadic first-order classical logic, Béziau also claimed that first-order classical logic is paraconsistent.

[2]Note that there are a number of *positive* developments that are aimed at generalizing the idea of Béziau. Those include Mruczek-Nasieniewska and Nasieniewski (2005, 2008, 2009) and Marcos (2005). In contrast, I will in part develop the same idea, but with somewhat different directions, motivated by some worries related to the idea.

[3]See, e.g. Berto (2015), Berto and Restall (2019) for a recent discussion in favor of modal account of negation.

[4]Note here that we are not assuming **L** to be an extension of the modal logic **K**. Instead, we are thinking of much weaker systems, possibly without any axioms or rules for the additional

vocabulary that are intended to capture modality.

[5]The understanding of modal logic here does not presuppose much, following the presentation by Lloyd Humberstone in (Humberstone, 2016).

[6]It is also known as **KD!**. For a detailed proof-theoretic treatment of $\mathbf{KDD}_c$, with an application to the revision theory of truth, see (Standefer 2018). The same system is also explored by Georg Henrik von Wright and Krister Segerberg under the name **W**. For the corresponding system that is definitionally equivalent to $\mathbf{KDD}_c$ in the language $\mathscr{L}$ is not yet known in the literature, to the best of my knowledge. One such system, for interested readers, is presented in the appendix.

[7]One of the interesting results that is established by Segerberg is that $\mathbf{KDD}_c$ is pretabular. According to Karpenko (2013, p.47), this was already known in 1979 as a result by N. M. Ermolaeva and A. A. Mučnik.

[8]If one is happy to let there be more than one negation in the system, then the very question might not be even addressed. This makes it clear that I am imagining that there might be a worry raised by those who think that there are at most one negation in the given system. Alternatively, one may prefer to say that a system is paraconsistent relative to this or that negation (I owe this last point to Jonas R. B. Arenhart). Further details, however, will be left for another occasion.

[9]Note that $\mathbf{KDD}_c\mathbf{B}$ is not completely without philosophical applications. See, e.g. De and Omori (2022), for an application of the system with the global consequence relation in the context of an alleged problem for semanticist discussed in Button (2016), Button and Walsh (2018).

[10]The observation here can be seen as a slight variation of a result established in Theorem 2 of Karpenko (2013). There are two things that are worth noting. First, instead of adding the symmetry, Karpenko adds the following condition (stated in our notation): for all $w_1, w_2, w_3 \in W$, $w_1 R w_2$ and $w_2 R w_3$ only if $w_1 = w_3$. Second, Karpenko refers to **FDE** with the boolean complementation as **Tr** in §5 of Karpenko (2013), and this is introduced in the context of considering a closely related system introduced by von Wright's truth logic. It is also observed that **Tr** has the Craig interpolation property, which is also proved in the first-order expansion in §5.1 of Kamide (2022) by Norihiro Kamide.

[11]Note here that the American plan semantics has been sometimes criticized by having too much freedom in combining the truth and falsity conditions for connectives (e.g. Berto and Restall 2019, §7, against De and Omori (2018)). However, $\mathbf{KDD}_c$ makes it clear that the involution condition for the star operation can be seen as a specific constraint among many other options. Therefore, the Australian plan semantics, considered via the star semantics, also have some freedom. For a specific case with a condition different from the usual involution condition, see Omori and Wansing (2022), Definition 54.

[12]Some readers may wonder what can be said for the case with quantifiers, given that Béziau claimed that first-order logic is also paraconsistent in (Béziau 2022). If we try to deploy the same strategy with the modal case, then we need to make sure to remove the principles that correspond to axioms T and $\mathrm{T_c}$, to this end, one may think of free logics. Further details, however, are left to interested readers.

[13]One may cast a doubt if the weakest possible system can be understood as a modal logic at all. A possible response is offered in Omori and Skurt (2024).

[14]Note that the system equivalent to **HYPE** was already introduced by Grigore Constantin

Moisil in 1942 (see Drobyshevich et al. (2022) for a detailed view of Moisil's work).

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Technical appendix

In this technical appendix, which is aimed at readers with technical interests, I will first present, model-theoretically, the modal logic $\mathbf{KDD}_c$ in the language $\mathscr{L}$. I refer to the system as $\mathbf{nKDD}_c$, for *negative* $\mathbf{KDD}_c$. I will then turn to present an axiomatic proof system and this will be followed by soundness and completeness results.

Definition 10. *A* functional interpretation *for* $\mathscr{L}$ *is a structure* $\langle W, *, v\rangle$ *where* $W \neq \emptyset$ *is a set of worlds,* $* : W \longrightarrow W$ *is a function, and* $v : W \times \mathsf{Prop} \longrightarrow \{0,1\}$*. The function* v *is extended to* $I : W \times \mathsf{Form} \longrightarrow \{0,1\}$ *as follows:*

$$I(w,p) = v(w,p), \qquad I(w,A\wedge B){=}1 \textit{ iff } I(w,A){=}1 \textit{ and } I(w,B){=}1,$$
$$I(w,\sim A){=}1 \textit{ iff } I(w^*,A){\neq}1, \qquad I(w,A\vee B){=}1 \textit{ iff } I(w,A){=}1 \textit{ or } I(w,B){=}1,$$
$$I(w,A\rightarrow B){=}1 \textit{ iff } I(w,A){\neq}1 \textit{ or } I(w,B){=}1.$$

Definition 11. *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \models A$ *iff for all functional interpretations* $\langle W, *, v\rangle$ *and for all* $w \in W$*, if* $I(w,B) = 1$ *for all* $B \in \Gamma$ *then* $I(w,A) = 1$.

Definition 12. *The system* $\mathbf{nKDD}_c$ *consists of the following axiom schemata and a rule of inference, where* $A \leftrightarrow B$ *abbreviates* $(A \rightarrow B) \wedge (B \rightarrow A)$.

(Ax1) $A \rightarrow (B \rightarrow A)$

(Ax2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

(Ax3) $((A \rightarrow B) \rightarrow A) \rightarrow A$

(Ax4)	$(A \wedge B) \to A$
(Ax5)	$(A \wedge B) \to B$
(Ax6)	$(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$
(Ax7)	$A \to (A \vee B)$
(Ax8)	$B \to (A \vee B)$
(Ax9)	$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
(Ax10)	$\sim(A \to A) \to B$
(Ax11)	$\sim\sim(A \to A)$
(Ax12)	$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(Ax13)	$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
(Ax14)	$\sim(A \to B) \leftrightarrow ((\sim A \to \sim(C \to C)) \wedge \sim B)$
(MP)	$\dfrac{A \quad A \to B}{B}$
(Contra)	$\dfrac{A \to B}{\sim B \to \sim A}$

Finally, we write $\vdash A$ *if there is a sequence of formulas* $\langle B_1, \ldots, B_n, A \rangle$ $(n \geq 0)$*, called* a derivation*, such that every formula in the sequence either (i) is an axiom of* $\mathbf{nKDD}_c$*, or (ii) is obtained by rules from formulas preceding it in the sequence. As usual, we define* $\Gamma \vdash A$ *as for some* $B_1, \ldots, B_n \in \Gamma$*,* $\vdash (B_1 \wedge \cdots \wedge B_n) \to A$*.*

Theorem 1 (Soundness)**.** *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash A$ *then* $\Gamma \models A$*.*

Proof. Tedious, but standard. I only note the following equivalences:

$I(w, \sim(A \to B))=1$ iff $I(w^*, A \to B) \neq 1$ iff $I(w^*, A)=1$ and $I(w^*, B) \neq 1$,

$I(w, \sim A \to C)=1$ iff $I(w, \sim A) \neq 1$ or $I(w, C)=1$ iff $I(w^*, A)=1$ or $I(w, C)=1$

It should then be easy to see that the case for (Ax14), which is probably the only unusual case, is also proved. □

Definition 13. *A set of formulas,* Σ*, is a* prime $\mathbf{nKDD}_c$-theory *iff (i)* $\mathbf{nKDD_c} \subseteq \Sigma$*, (ii) it is closed under* (MP)*, (iii)* $A \vee B \in \Sigma$ *implies* $A \in \Sigma$ *or* $B \in \Sigma$ *and (iv) it is* non-trivial*, namely if* $A \notin \Sigma$ *for some* A*.*

The following lemma is well-known, and thus the proof is omitted.

Lemma 14. *If* $\Gamma \nvdash A$ *then there is a prime* $\mathbf{nKDD}_c$*-theory,* Δ*, such that* $\Gamma \subseteq \Delta$ *and* $\Delta \nvdash A$*.*

Note also that we obtain the following lemma.

Lemma 15. *If* Γ *is a prime* $\mathbf{nKDD}_c$*-theory, then* $A \to {\sim}(B \to B) \in \Gamma$ *iff* $A \notin \Gamma$.

Lemma 16. *If* Γ *is a prime* $\mathbf{nKDD}_c$*-theory, then* $A \to B \in \Gamma$ *iff* $A \notin \Gamma$ *or* $B \in \Gamma$.

Lemma 17. *Let* Γ *be a prime* $\mathbf{nKDD}_c$*-theory, and let* Γ^* *be defined as follows:*

$$\Gamma^* := \{A : {\sim}A \notin \Gamma\}.$$

Then Γ^* *is also a prime* $\mathbf{nKDD}_c$*-theory.*

Proof. For $\mathbf{nKDD_c} \subseteq \Gamma^*$: suppose $C \in \mathbf{nKDD_c}$. Then, by (Ax1) and (Contra), we obtain ${\sim}C \to {\sim}(A \to A) \in \mathbf{nKDD_c}$ and therefore ${\sim}C \to {\sim}(A \to A) \in \Gamma$, and in view of Lemma 15, ${\sim}C \notin \Gamma$. But this is equivalent to $C \in \Gamma^*$, as desired.

For MP:

$$\begin{aligned} A \in \Gamma^* \text{ and } A \to B \in \Gamma^* &\text{ iff } {\sim}A \notin \Gamma \text{ and } {\sim}(A \to B) \notin \Gamma \\ &\text{ iff } {\sim}A \notin \Gamma \text{ and } ({\sim}A \to {\sim}(C \to C)) \notin \Gamma \text{ or } {\sim}B \notin \Gamma) \\ &\text{ only if } {\sim}B \notin \Gamma \\ &\text{ iff } B \in \Gamma^* \end{aligned}$$

For primeness:

$$\begin{aligned} A \vee B \in \Gamma^* &\text{ iff } {\sim}(A \vee B) \notin \Gamma \\ &\text{ iff } ({\sim}A \wedge {\sim}B) \notin \Gamma \\ &\text{ iff } {\sim}A \notin \Gamma \text{ or } {\sim}B \notin \Gamma \\ &\text{ iff } A \in \Gamma^* \text{ or } B \in \Gamma^*. \end{aligned}$$

For non-triviality: if ${\sim}(p \to p) \in \Gamma^*$, then ${\sim}{\sim}(p \to p) \notin \Gamma$, but this will contradict that ${\sim}{\sim}(p \to p) \in \Gamma$. This completes the proof. □

Theorem 2 (Completeness)**.** *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models A$ *then* $\Gamma \vdash A$*.*

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash A$. Then by Lemma 14, there is a $\Pi \supseteq \Gamma$ such that Π is a prime $\mathbf{nKDD}_c$-theory and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle X, *, I \rangle$, where $X = \{\Delta : \Delta \text{ is a prime } \mathbf{nKDD}_c\text{-theory}\}$, $\Sigma^* = \{A : {\sim}A \notin \Sigma\}$, and I is defined thus. For every state, Σ and propositional parameter, p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma.$$

We show that the above condition holds for any arbitrary formula, B:

$$(*) \qquad I(\Sigma, B) = 1 \text{ iff } B \in \Sigma.$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models A$. The proof of $(*)$ is by an induction on the complexity of B.
For negation: the standard proof carries over from the case in **FDE**.

$$\begin{aligned} I(\Sigma, \sim C) = 1 & \text{ iff } I(\Sigma^*, C) \neq 1 & \\ & \text{iff } C \notin \Sigma^* & \text{IH} \\ & \text{iff } \sim C \in \Sigma & \text{By definition of } \Sigma^* \end{aligned}$$

For disjunction: We begin with the positive clause.

$$\begin{aligned} I(\Sigma, C \vee D) = 1 & \text{ iff } I(\Sigma, C) = 1 \text{ or } I(\Sigma, D) = 1 & \\ & \text{iff } C \in \Sigma \text{ or } D \in \Sigma & \text{IH} \\ & \text{iff } C \vee D \in \Sigma & \Sigma \text{ is a prime theory} \end{aligned}$$

For conjunction: Similar to the case for disjunction.
For implication: this is also quite standard.

$$\begin{aligned} I(\Sigma, C \to D) = 1 & \text{ iff } I(\Sigma, C) \neq 1 \text{ or } I(\Sigma, D) = 1 & \\ & \text{iff } C \notin \Sigma \text{ or } D \in \Sigma & \text{IH} \\ & \text{iff } C \to D \in \Sigma & \text{Lemma 16} \end{aligned}$$

Thus, we obtain the desired result. □

As a corollary, we obtain the soundness and completeness results for **BD+**, a result already observed in Omori and Skurt (2019 §6.1), and explicitly stated in Niki and Omori (2024).