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SOME REMARKS ABOUT GOING TOWARDS INCONSISTENCIES[1]

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Abstract. Inconsistencies! What do they mean? Can we support them? With this paper, we hope to contribute to the claim that we can tolerate inconsistencies in certain situations even without considering any logic that may enable us to do that, say some paraconsistent logic. We argue that in many cases where we apply reason we work in domains where inconsistencies appear and even so we neither get them out (but 'support' them) nor modify the underlying logic (such as classical logic) to avoid logical troubles. To make things more precise, we distinguish between *inconsistency*, *anomaly*, and *contradiction*. Our thesis is that we can reason sensibly even with classical logic in the presence of inconsistencies once (as we explain) we either 'do not go there' or make things so that the inconsistencies cannot be joined to arrive at a contradiction. Some sample cases are given to motivate the discussion.

Keywords: inconsistency • contradiction • anomalies • paraconsistency • complementarity • quantum negation • Zande logic

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> "[O]ne may realize that one is (and for some time will be) unable to resolve an inconsistency within some domain, but nevertheless aim at solving another problem within that domain."
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> Joke Meheus 2002, p.151

1. Introduction

Inconsistencies are situations, states of affairs, sentences, whatever, which conflict and potentially go against one another. Inconsistencies may exist even if not noticed by the scientist, who in general continues to work with her theory as if everything is fine. When inconsistencies are noticed, we call them *anomalies*, since 'now' the scientist is aware of the inconsistencies and (in principle at least) should be able to justify her attitude towards them. But, in many cases, the anomalies are not solved and even so, the *paradigm* continues to hold, as supported by Thomas Kuhn (1996)) (we shall be back to Kuhn below).

For instance, the Infinitesimal Calculus (Newton's Method of Fluxions and Leibniz's account) was used by many as a good theory before the knowledge of the inconsistencies that may arise from the hypotheses of the infinitesimals, as noticed by George Berkeley in 1734 (Berkeley 1734). As realised by Berkeley, a quantity o is sometimes taken as not zero (when seen as an increment) and sometimes as zero when "[in the] expression $x + o$, o stands for nothing" (op.cit., for instance, at XIII). When the infinitesimals became anomalies, the solution was to modify the theory, which happened with the introduction (by Cauchy and others) of the famous epsilon and delta definitions of limit, derivatives, and integrals.[2]

Another relevant example comes from naïve (Cantorian) set theory. As proposed by Georg Cantor at the end of the XIXth century, set theory was a useful theory for founding mathematics, since all known mathematical concepts such as natural numbers, functions and the like could be written in terms of sets. But soon, several inconsistencies (in the form of paradoxes) arose in the field, such as Burali-Forti's, Cantor's himself, and several others. When these 'paradoxes of set theory' appeared, it was noticed that the theory involved inconsistencies.[3] As Fraenkel, Bar-Hillel and Levy explain, the core of mathematical activity, such as analysis and geometry, were not (at first glance, for sure) affected by the rising of the (noticed then) anomalies; as they say, "they appear chiefly in a region of extreme generalisation, beyond the domain in the concepts of these disciplines are actually used." (Fraenkel et al. 1973, p.4) That is, until the inconsistencies and even the anomalies bring real problems, namely, that the field is contaminated by the problem of the contradictions, the scientist continues to work in the known paradigm. As is known, in set theory things became worse when paradoxes such as Russell's appeared, since they made use of quite very basic assumptions such (in the case) that any given property gives rise to a set of those objects that satisfy the property. When things go to this point, something must be done. In the set-theoretical case, it conduced to the axiomatisation, which was achieved and resulted in different set theories (Fraenkel et al. 1973).

A third situation is Bohr's theory of the (Hydrogen) atom. Ever since its statement it was noticed that the theory involves incompatible suppositions, namely, that electrons orbit in stable orbits around the nucleus and that (according to Maxwell's electrodynamics) in orbiting they should emit energy and collapse into the nucleus, something not observed. Although some guess that the theory involves contradictions, this is disputable (see Vickers 2013); anyway, inconsistency between the hypotheses is clear. The theory was superseded by the more advanced view in terms of probability densities: there are no 'orbits', but just regions occupied by the electrons. But Bohr's notion of complementarity and the present-day Standard Model of particle physics are more interesting, as we shall see below.[4]

A fourth example comes from social behavior, from an analysis of the Zande people. It does not involve a scientific theory strictly speaking but constitutes a good example of a situation where inconsistencies are noticed (by a Western philosopher at least) but the Zande people do not consider them as anomalies. We shall analyze this case below. Zande people seem to follow Wittgenstein's claim that sometimes we can live with inconsistencies once we 'do not go there', that is when we do not turn them into anomalies. We shall explain this point below.

There are plenty of examples of situations involving inconsistencies and even anomalies in the literature, and it would be a huge work (if not impossible) to mention all of them here. Disagreements appear practically in any field, such as when studying the solar system (Ptolemy and Copernicus disagree), the nature of light (Newton versus Huygens), evolution (Lamarck versus Darwin), and so on. We cannot mention all situations, but the above cases seem enough to exemplify our claims, and our references provide more places where the discussion can be found.

Anomalies, when noticed and when the underlying logic of the theory is well delineated, can originate contradictions.[5] But this makes sense only in a well-defined language and when a well-defined logic is being considered. In this sense, by a contradiction, we understand the conjunction of a formula with its negation, something of the form '$\alpha \wedge \neg\alpha$', being α a formula, $\neg$ the symbol for negation and $\wedge$ the symbol for conjunction. Sometimes the logic is not adjunctive, so we cannot form the conjunction of two arbitrary formulas, but even so the presence of *contradictory formulas*, one being the negation of another, causes problems due to the (usually accepted) Explosion Rule (or Scotus Principle), $\neg\alpha \to (\alpha \to \beta)$, which is a theorem of most logic systems. In an informal discourse, we can arrive at what could be called a *pseudo* contradiction, something that resembles a formal contradiction in the form stated above, yet they are also enough to cause foundational problems in most cases since there are no well-defined language and logic.[6]

Following these lines, we wish to present some ways in which we can live with inconsistencies and even with anomalies and contradictory situations. Summing up, they are the following, whose details will be presented in the next sections:

(1) We can get along with inconsistencies when we are not conscious about them, as the forerunners of the Infinitesimal Calculus did, or then the set-theorists before the rise of the paradoxes and perhaps many other situations not revealed yet; who knows if a Grand Unified Theory of particle physics or even Quantum Mechanics and the General Relativity are syntactically consistent?[7]

(2) We can also acknowledge that there are inconsistencies but refuse to go there, living in the 'consistent part' only. This raises two sub-cases:

   (i) The first alternative is to adopt a rather naive and pre-scientific attitude:

to acknowledge the inconsistencies but ignore that they are anomalies and live happily. We call this *the Zande way* to be seen soon.

(ii) Maybe you are following some form of constructive philosophy so despite you realizing that α and β (the contradictory sentences, β being equivalent to the negation of α) are possibilities, you have not got yet a way to know which is the case (by supposing that you are using some logic that obeys the Explosion Principle). So, the scientist *suspend the judgment* about the validity or the invalidity of the cases and continue to work until the progress of science enables her to make a choice. I call this alternative *the Wittgenstein way* to be justified below.

(3) We can also acknowledge the existence of inconsistencies but suppose that they are useful and relevant so that one should tolerate them. This also raises two sub-cases:

(i) One tries to find a new logic for her theory, surely some form of paraconsistent logic (as said before, inconsistent-adaptive logics enter this realm). As said above, in this case, one will be faced with the difficulty in explaining what should be a paraconsistent negation; as said before, there is still a lack in the philosophical literature about the 'real meaning' of a paraconsistent negation; I mean: when we say that a paraconsistent logic enables one to accept contradictory formulas without trivialisation, what does one mean by 'not'-α?

(ii) We can 'support' both α and β as inconsistent but avoid the situations where they are put together. This is typical of Bohr's complementarity principle and (apparently) of the Grand Unified Theory (GUT) of particle physics (see below). Notice that this case is distinct from (2) above. Now one *do not want* (Bohr) or can't (GUT) go to the conjunction of two contradictory (we prefer to call them 'complementary') sentences or situations, and not simply ignore that some logical trouble can arise (perhaps due to the ignorance of the mechanisms of logic, as in the Zande case).

Let us see the reasons we see for suggesting the above distinctions.

2. Ignorance about inconsistencies and when they turn into anomalies

The first case is the acknowledged ignorance about inconsistencies are the Infinitesimal Calculus in its beginnings. It is remarkable that the Marquis de L'Hospital, in

1696, in the first book ever written about Calculus, wrote the following:

> One wonders who can take indifferently from each other two quantities which differ only by an infinitely small quantity: or (which is the same thing) only a quantity which is not increased or decreased that by another quantity infinitely less than itself, can be entrusted as remaining the same. (L'Hospital 1696, Book I, pp.2–3)

There is a clear *inconsistency* here since the Marquis is taking as being 'the same' two quantities which differ, even if by an infinitesimal quantity, going against the standard way to understand what 'the same' would mean. It would be a problem for the historian of mathematics to try to unravel the fact whether he (and others as well) have noticed the inconsistency. Since Calculus continued to be used, we suggest that they did not realize that a contradiction could appear, that is, that their use of infinitesimals was an anomaly.

Even with hypotheses of this kind, it is agreed that Calculus was useful in several areas such as in physics, as shown by Newton himself and by many others, such as the Marquis of L'Hospital and his analysis of infinitesimals for the analysis of curves (L'Hospital 1696). It took a lot of time to 'correct' Calculus with the introduction, in the XIXth century, of the well-known definitions using the famous epsilons and deltas.[8] The theory was useful insofar as one does not go to the contradiction. Even today it is not uncommon to see an engineer speaking in 'an infinitesimal element of volume' using differentials: $dV = dxdydz$ even without the delicate consideration of the situation through Mathematical Analysis; they work almost intuitively within the spirit of the ancient calculus.

The situation can be exemplified this way. Suppose that in a Calculus previous to the definition of limit we have the real function $y = x^2$ and that we wish to find its derivative, namely, the quotient $\Delta y/\Delta x$. Let us suppose that we give an *infinitesimal increment* Δx to x so that y gains also an increment Δy.[9] Then we have now $y+\Delta y = (x+\Delta x)^2$, hence $y+\Delta y = x^2+2x\Delta x+(\Delta x)^2$. Hence, cutting y and x^2 since they are equal, we get $\Delta y = 2x\Delta x+(\Delta x)^2$. By dividing both members by Δx, which is supposed to be not zero since it is an 'increment', we get $\Delta y/\Delta x = 2x+\Delta x$. But the increment, being infinitesimal, can be dispensed with, so we arrive at the derivative of the function, that is, $\Delta y/\Delta x = 2x$.

Of course, this reasoning is fallacious since we have assumed both that the increment Δx is not zero and also that it is zero (when it is ignored). The use of infinitesimal by Newton and Leibniz was inconsistent as largely documented, but even so, the Calculus provided a great service.

Set Theory is our second example. As proposed by Georg Cantor by the end of the XIXth century, set theory is inconsistent, but he and others did not realize that until the paradoxes appeared; you can find reference to them in Fraenkel et al. (1973). As with the case of Calculus, at the beginning the inconsistency do not turn into anomalies; when these were acknowledged, mathematicians proposed different axiomatizations of the notions of sets and classes, resulting in different and no equivalent 'set theories'. The just given reference is a good guide.

There are basically two attitudes facing situations like those above. If one is a mathematical realist, she believes that one of the inconsistent statements, say of an open problem in mathematics, is true and that maybe we need further axioms to decide. The other alternative comes with a constructive-like approach: we suspend the judgment until someone has written symbols sufficiently enough to show which one of the rival statements is the case. Both are used in a philosophical setting.

Another case is mentioned by Imre Lakatos and concerns informal geometry. Suppose we are trying to see the relation among the number F of vertices, E of edges and F of faces of polyhedra. In regular polyhedra, such as a cube, we easily arrive at Euler's relation $V-E+F=2$, and there is even a proof of this fact for regular polyhedra given by Cauchy. The conjecture is: does this fact apply to all polyhedra? In assuming that, we easily get a contradiction; it is enough to consider a polyhedron (a 'hollow cube') formed by a pair of nested cubes, that is, a pair of cubes one of which is inside another, but does not touch it Lakatos (1976, p.13). Clearly $V-E+F=4$ in this case. Of course, the mathematician knows that by using the transitive law of identity, one arrives at $2=4$, which contradicts the mathematical fact that $2 \neq 4$. But we need logic to do that. On the contrary, in the quantum mechanical examples given earlier, no 'deduction' can be made showing that the inconsistencies lead to contradictions since we can't do both the 'particle' and the 'wave' experiments at once.[10]

A final example comes from quantum physics. It shows how physicists, to get better results ('FAPP', "for all practical purposes", in the words of John Bell), assume inconsistent claims which, although known in principle, are ignored for practical purposes. The case is this: when we have two quantum systems whose states are described by wave functions (or state vectors) so that we can *provisorily* consider them as isolated systems if they are sufficiently apart (this is linked to the so-called *de Broglie wavelength* of the systems). In this case, all happens as if they were two distinct and isolated entities behaving according to classical physics. Suppose we have two elementary particles of the same kind located at different points A and B. Being x_1 and x_2 their coordinates, let $\psi_A(x_1)$ and $\psi_B(x_2)$ be the wave-functions of the particles. Then the joint probability amplitude for finding the first particle at x_1 and the second at x_2 might be thought as being done by the tensorial product $\psi_A(x_1)\psi_B(x_2)$, but it is not so since this would conflict with the Indistinguishability Postulate, which states that for all vectors (states) $|\psi\rangle$, all operators $\hat{A}$, and all particle label permutation operators P, we must have $\langle\psi|\hat{A}|\psi\rangle = \langle P\psi|\hat{A}|P\psi\rangle$, that is, the expectation values are the same before and after a permutation. If the particles are indiscernible, nothing different would get if they are exchanged, that is that the joint probability amplitude would be $\psi_A(x_2)\psi_B(x_1)$, which is *different* from the first product once the tensor product is not commutative. The right vector for describing the joint probability amplitude is

$$\psi_{12} = \frac{1}{\sqrt{2}}\big(\psi_A(x_1)\psi_B(x_2) \pm \psi_A(x_2)\psi_B(x_1)\big),$$

where the plus sign holds for bosons and the minus sign holds for fermions. The joint probability density is then given by

$$\|\psi_{12}\|^2 = \frac{1}{2}\big(\|\psi_A(x_1)\|^2\|\psi_B(x_2)\|^2 + \|\psi_A(x_2)\|^2\|\psi_B(x_1)\|^2 \\ \pm 2Re\,\langle\psi_A(x_1)\psi_B(x_2)|\psi_A(x_2)\psi_B(x_1)\rangle\,\big),$$

where the last term $2Re(\ldots)$ is the *interference term*. This term, for the 'practice of physics', can be eliminated, since the overlap of the two wave functions becomes appreciable only when the distance between the particles is not much larger than the de Broglie's wavelength. As Dalla Chiara and Toraldo di Francia emphasise,

> This is the reason why an engineer, when discussing a drawing, can *temporarily* make an exception to the anonymity principle[11] and say for instance: 'Electron a issued from point S will hit the screen at P while electron b issued from T hits it at Q.' But this mock individuality of the particles has very brief duration. When the electron hits the screen (...) it meets with other electrons with substantial overlapping, and the individuality is lost. In fact the de Broglie wavelength of an electron inside an atom is on the same order of magnitude as the atomic diameter. (Dalla Chiara and Toraldo di Francia 1993)

This shows that the supposition that the interference term can be neglected is similar to the supposition that infinitesimals can be dispensed with. The results in quantum physics, so as those in the Calculus, are right (as far as we know), but from the logical foundational point of view the logical inconsistency is evident.

So, when inconsistencies are noticed, we have anomalies. An anomaly is, according to the Oxford Languages Dictionary, something that *deviates* from the standards, for instance, the deviation of some measurement from its expected value. In the case of inconsistencies, the deviation is mainly from the atavistic concept of consistent theories, which became a paradigm in mathematics and even in science at least since Aristotle, who in the Book Γ of his *Metaphysics*, regarded the Principle of Non-Contradiction as the most certain of all principles. A theory is (syntactically) consistent if it does not derive two contradictory sentences. Sometimes the anomalies are detected in the form of 'conflicts' within a certain theory; the American Physical Society reports that in particle physics, an anomaly is an experimental result that *conflicts* with the Standard Model, "but fails to overdrum it for lack of sufficient evidence."[12] There are also situations when a theory makes a prediction which does not conform to observations; for instance, Newton's mechanics had problems in predicting the motion of the moon's apogee. This is remarkable; despite the existence of anomalies, the scientist continues to use the theory until the anomaly *really* turns into a problem, say by showing a contradiction. Thomas Kuhn believed that the *paradigm* (the 'normal' science) usually involves anomalies and even so they are not (in principle) discharged. In these cases, at least in the standard science we practice, something must be done, and several philosophers have proposed mechanisms for 'correcting' (or 'reconstructing') the theories (enough to mention Popper and Lakatos; see below).

For Thomas Kuhn, an anomaly is a *violation* of the paradigm-included expectation that governs normal science. In his opinion, anomalies provide the impetus for a paradigm change (Kuhn 1996), but the paradigm may resist for some time before being exchanged by a new one. For Larry Laudan, science is fundamentally a problem-solving activity (Laudan 1981), so the aim of science would be to maximize the scope of solved empirical problems while *minimizing* the scope of anomalous and conceptual problems, but we could add, not necessarily eliminating them.

3. Acknowledging inconsistencies

In acknowledging that inconsistencies do exist, as anticipated above, led us to two alternatives posed as (2i) and (2ii). Let us begin with the second one. A certain theory is such that there is some statement to which the scientist does not have *yet* an answer about its validity or invalidity. Their conjunction is impossible under classical negation (contradictoriness), and something like a mathematical conjecture will be either true or false, not both. Examples come from any open problem in mathematics. This case can be exemplified by some considerations made by Ludwig Wittgenstein.

Wittgenstein has made some remarks concerning possible inconsistencies in arithmetics.

> If a contradiction were actually found in arithmetic, this would show that arithmetic with such a contradiction can serve us very well. (Wittgenstein 1967, p.xvii) (see also p.181)

It should be realised that Wittgenstein's conception of mathematics is very peculiar, and we would not generalise and quote him arbitrarily. But taking into account that for him "arithmetics is not based on logic", but on general 'principles' that he suggests can be taken as 'laws of thought' (Diamond 1976, p.272) so as that "the rules of logical inference are rules of the *language game*" (Wittgenstein 1967), we may guess that a mathematical theory, and in particular arithmetics, is never finished, which is also corroborated by his reference to Hardy, the English mathematician: "Professor Hardy says, 'Goldbach's theorem is either true or false.' — We simply say the road hasn't been built yet." (Diamond 1976, p.138) (this is the situation we have referred to above as a case where symbols were not written in a sufficient number yet). That is, arithmetics, and hence mathematical theories, in general, are always open to novelties, and so cannot be taken as completely axiomatized or formalised. Notice that we have said 'novelties' and not something like 'new results', since it is clear that an axiomatized/formalized theory can give us 'new results', for instance, new theorems. The discussion of whether these new theorems are already encapsulated in the axioms is not to be discussed here.[13] What we guess is that words like 'completeness' of a theory (in Gödel's sense) seem do not make sense to him.[14] For instance, Wittgenstein has also claimed that

> Can one find a contradiction in a certain system? One might say, "It depends on you." —One might say, "Finding a contradiction in a system, like finding a germ in an otherwise healthy body, shows that the whole system or body is diseased." —Not at all. The contradiction does not even falsify anything. Let it lie. *Do not go there*. (Diamond 1976, p.138) (my emphasis)

I tend to agree with him, mainly with respect to his final advice: if some inconsistency appears, 'do not [necessarily] go there'! In the empirical sciences, and even in informal mathematics, that is, when mathematics is still in its heuristic development (Lakatos 1976), the mathematician may face inconsistencies which (if in principle she is not a paraconsistent mathematician) suggests her to revise her stuff in order either to eliminate the problem or to circumvent it, as did Bohr with his ideas on complementarity, as we shall see soon. Only when the system is formalized and if it is formalized using classical logic or one of the most known logical systems (that is, one which assumes the Explosion Rule), the presence of inconsistencies may bring problems to the theory.

A case in physics may be this one. The search for the unified gauge group of the Standard Model of particle physics is still an open problem. A GUT (Grand Unified Theory) would unify three of the four fundamental forces of nature, the electromagnetic force, the weak force and the strong force; only gravity is left out due to the lack of knowledge of how to add it to the schema. The first two forces are unified in the Electroweak Theory, whose gauge group is $U(1) \times SU(2)$. (It is not relevant here what these things mean specifically.) The strong force is dealt with by Quantum Chromodynamics (QCD), whose gauge group is $SU(3)$. The Standard Model joins the two theories, but they do not work together. A GUT would be a unified theory, having as its gauge group what is written as $U(1) \times SU(2) \times SU(3)$ which would encompass a unified force, not observed yet (according to the Wikipedia entry 'Grand

Unified Theory', Sept.25, 2023). This means that the Standard Model works in two separate parts: the Electroweak Theory and QCD. This is known, but physicists are trying yet to find the right gauge group for a GUT some name $SU(5)$ (Glashow 1991). Notice that in this case, physicists are completely aware of the present-day difficulty of joining the three forces in a GUT despite the subject being debatable. They do not go there because they don't know how.

Anomalies can remain in a theory and the scientist may be aware of them, and even so, the theory is not put aside. Popper claimed that in such cases the scientist would choose between two alternatives, namely, either to change the theory or to propose some 'auxiliary hypothesis' which can give her ways of testing predictions (Popper 2002). But it is generally agreed that in general scientists do not work this way; as Kuhn has suggested, they usually continue in the paradigm (the theory they are normally using) even in the presence of anomalies; they do not reject a paradigm without having at hands a substitute:

> The decision to reject one paradigm is always simultaneously the decision to accept another, and the judgment leading to that decision involves the comparison of both paradigms with nature *and* with each other. (Kuhn 1996, p.77 (emphasis in the original)

So, in a broad context, say in a huge database, where pieces of information are being stored, one might not realise that inconsistencies were introduced and only later the 'problems' can be detected, *once we go there*. Precisely for this reason paraconsistent logics were introduced in computer sciences dealing with inconsistent information, since a computer possibly 'may go there' when running over the collected data (see Blair and Subrahmanian (1988) for the presentation of a kind of paraconsistent logic that can be used in computer science to ground expert systems involving inconsistent data). This is an interesting case in applications. In getting data to account for information in a field of knowledge, the scientist may introduce conflicting facts, perhaps due to the discrepancies among the scientists themselves and under the hypothesis that the data do not come from just one of them. So, a program which can make inferences, if running over such a database, may encounter conflicting things and if the logic is not paraconsistent, the system becomes trivial. The consequences of such a fact are studied in paraconsistent engineering (Akama 2016).

The case (2i), where inconsistencies are acknowledged but do not turn into anomalies is more difficult to find in science. But just to illustrate, let us mention the case of witchcraft among the Azande.

4. The Zande case

Here is a typical case where someone *does not go there*, to use Wittgenstein's terms, but not because science has not evolved sufficiently, but because the sequence of some inference that would lead to a contradiction is stopped when one wants so that she keeps with the partial conclusion only, which is suitable for her. This is very well documented in the case of the Azande, a people who lived in Central Africa between the rivers Nile and Congo (more in today South Sudan), as related by the anthropologist E. E. Evans-Pritchard in his book *Witchcraft, Oracles, and Magic Among the Azande* (Pritchard 1976).[15] Summing up to go to our point, the fact is that the Azande believe in witchcraft; according to them, some people are witches and these can injure other people by a psychic act. They distinguished between witches and sorcerers, and someone can consult an oracle to know if someone else is injuring him. If the oracle says yes, then the suspect is a witch, but if he says no, nothing is inferred in this sense. It seems that they do not see that either someone is a witch or is not so that if not one, then another possibility (the deductive rule $p \vee q, \neg p \vdash q$, known already by the Stoics), but perhaps the better interpretation would be that they *refuse* to proceed to the conclusion (see below). To the Azande, witchcraft is a substance in the body of the witch in which various small objects can be found, and this substance is sometimes extracted by autopsy (Evans-Prichard reported that he never saw an exemplar). According to their beliefs, the substance is inhered and transmitted by unilinear descent from male to male; the son of a witch is a witch, but the daughters are not. A clan is formed by direct descent and not by adoption.

The interesting fact is that standard reasoning would say that if someone is found to be a witch, then the whole clan is formed by witches, but as Evans-Pritchard says, "[the] Azande see the sense of this argument but they do not accept its conclusion, and it would involve the whole notion of witchcraft in contradiction were they do so". He continues: "Azande do not perceive the contradiction as we perceive it because they have no theoretical interest in the subject and those situations in which they express their beliefs in witchcraft do not force the problem upon them." (ibid, p.3) The argument could be put in the following terms (in our present-day way of reasoning) (Jennings 1989):

(1) All and only witches have witchcraft substance.

(2) Witchcraft-substance is always inherited by the same-sexed children of a witch.

(3) The Zande clan is a group of persons related biologically to one another through the male line.

(4) Man A of clan C is a witch.

(5) The conclusion, not considered by the Azande: Everyman in clan C is a witch.

The Azande stopped the supposed deduction, not going to its end, that is, they did not go to step (5). One of the reasons is that in these cases, they regard the witch as not 'really' being part of the clan, but that he is a bastard, the son of another father, introduced in the clan artificially by a sin of the mother, who now will suffer for that even if this is a false accusation (according to our standards). This is just a way to 'save' the clan by introducing another hypothesis in the deduction, in a typical non-monotonic reasoning. But even without introducing new hypotheses (recall that this is one of Popper's alternatives), similar forms of reasoning appear among us in our daily lives and even in the practice of science; we continue with our hypotheses until we get something we wish, trying to solve the problems we face even in the presence of inconsistencies, so 'ignoring' that the continuation turns them into anomalies which may lead them to a contradiction and sometimes finding some excuse to them. By the way, perhaps the Azande did not know the notion of 'contradiction' as we do and surely they don't care about the laws of 'our' logic. The Zande paradigm is preserved even in the presence of such inconsistencies.

The presence of inconsistencies does not necessarily give us a contradiction if *we do not go there*, to use Wittgenstein's claim. We and the scientist may work in frameworks involving inconsistencies but we may avoid obtaining a contradiction and do not always change the framework. If, and let us emphasize the word 'if' we are to make the underlying logic explicit, for sure we will need to assume some form of paraconsistent logic, even if in the form of the non-adjunctive paraclassical logic (see section 5 below).

This way of reasoning entails that one can understand a theory (or a 'paradigm') not as a whole, but in a species of constructive thing, despite the underlying suppositions (or 'logic') that may conform to classical settings. That is, a theory may contain inconsistencies not perceived yet although the reasoning is 'classical' in the sense of accepting the Explosion Rule; the *metatheory* looks constructive, yet the theory itself is not. This is similar to Bourbaki's way of seeing mathematics; according to him, an open conjecture is either true (it has a proof) or false (its negation has a proof), but ignorance would be due to the fact that one has not written enough symbols yet (that is, we have neither a proof of the conjecture nor of its negation); see da Costa and Krause (2020). This is quite similar to what we have termed 'Wittgenstein's way'.

5. Do not put inconsistencies together

A different situation comes when the inconsistent suppositions are acknowledged but are not put together. In quantum mechanics, in 'Schrödinger's picture', the state of a quantum system evolves deterministically according to Schrödinger's equation (SE), but when a measurement of some observable is made, this kind of evolution is completely modified: the evolution is no longer deterministic. Roger Penrose speaks about two 'evolution procedures' of the states, the **U** procedure, which follows SE and the **R** procedure which, as he says, "change the rules" by the 'reduction' (or the collapse) of the quantum state (Penrose 1999, p.323). Notice that the two procedures do not occur together, so they do not conduce to a contradiction since their conjunction is never observed or postulated by the theory. A similar conclusion happens in situations exemplified by the 'paradox' of Schrödinger's cat, well-known in the literature so it is not necessary to revise it here. The important point is that *before measurement* the state of the cat is a superposition, so the states 'cat alive' and 'cat dead' are sharing an entangled situation and none of them is the case, much less both as usually guessed in popular books. The states 'cat alive' and 'cat dead' do not hold at once; the superposed state is *another state*; see (Krause and Arenhart 2016).

Another example could be the assumed fact that a quantum system may behave as if it were a 'classical particle' when sent to an apparatus for instance in the two-slits experiment when one of the slits is closed, or like a wave when the two slits are open, so contributing to the production of interference patterns. The fact is that 'particle' and 'wave' are incompatible descriptions of a system, being inconsistent with one another, but cannot be put together in a single description, yet need to be both taken into account for the full understanding of the phenomenon. Again, they do not produce a contradiction. As we have seen before, a contradiction, to be stated, requires that some logical notions are made explicit and that we can 'put together' the contradictory situations. If logic is not used, the most we can say is that we are facing anomalies.

The last example is one of the most interesting cases where inconsistencies appear but are not put together; it involves Bohr's notion of *complementarity* in quantum physics (see Bohr 1985). It is agreed that Bohr's notion is difficult to understand and much discussion has been produced on its meaning ever since it was stated; see Jammer (1966; 1974), da Costa and Krause (2006). Anyway, it is agreed that Bohr has considered two inconsistent notions without claiming that they conduce to a contradiction, one of them being the above-mentioned particle and wave aspects of a quantum system. They are conciliated by his Principle of Complementarity; he assumes the possibility of

> The existence of different aspects of the description of a physical system, seemingly incompatible but both are needed for a complete description of the system. In particular, the wave-particle duality. (French and Kennedy 1985, p.370)

To give an interpretation to this notion, we assume the following: we say that two sentences (propositions, formulas, whatever you prefer to call them) α and β *complementary* if and only if there exists a third formula γ such that from α we derive

γ and from β we derive the negation of γ, that is, $\neg\gamma$. The fact is that even if we have (not yet) derived the two contradictory formulas, we can work with complementary sentences in our theory and in some cases, as those pointed out by Bohr, we *need* to use them to achieve to a full understanding of a phenomenon. The particle and the wave aspects of a phenomenon must be both taken into account in orthodox quantum mechanics, yet they are never seen together at the same time, or in the same experiment.

Complementary sentences are inconsistent with one another (surely a formula and its 'classical' negation are complementary in terms of the given definition), and we usually work with such sentences in the same context. Bohr of course was not occupied with logical questions, but if we are to accept both complementary sentences and wish to avoid that they give us a contradiction, then some form of logic involving a distinct notion of deduction can be useful. For instance, *paraclassical* logic may be such a system; let us see it at a propositional level. The 'paraclassical deduction' $\vdash_P$ is so that if Γ is a set of premises and α is a formula, then we say that $\Gamma \vdash_P \alpha$ (α is deduced paraclassically from Γ) iff (1) α is a classical thesis, that is, a theorem of classical propositional logic or (2) there exists a subset $\Delta \subseteq \Gamma$, consistent from the 'classical' point of view,[16] so that $\Delta \vdash \alpha$, being '$\vdash$' the deduction in classical logic. The language of paraclassical logic is exactly similar to that of classical propositional logic.

So, from a set of premises of the form $\Gamma = \{\ldots, \alpha, \ldots, \beta, \ldots\}$ such that for some formula γ we have that $\alpha \vdash \gamma$ and $\beta \vdash \neg\gamma$, we cannot deduce the contradiction $\gamma \wedge \neg\gamma$ paraclassically.[17] The sentences or formulas α and β can continue to form part of the context without entailing a contradiction.[18]

6. The logic and the reasoning of the 'not going there'

One of the referees of this paper guessed that I should be clear about the used expression 'do not go there' attributed to the Azande and perhaps to other people such as Wittgenstein, since according to him/her, I was not very precise in using the expression. The case is that although being advised that anomalies are present, which could lead to contradictions, some people, like the Azande, simply refuse to discuss the issue, preferring to stop a possible chain of reasoning and not go to the anomaly; the advice would be this (given in our Western way of reasoning): if you thought about the 'complete' inference, you shall realize that a contradiction follows. But the fact is that the anomaly occurs in our perspective, not theirs. Others, such as probably Wittgenstein and perhaps most empirical scientists, recognize the possible contradiction but prefer to work in other parts of the theory, leaving, so to say, the anomalies confined in a way that they do not force them to face contradictions. It is in this sense that we have used the expression 'not going there': not going *to the contradiction*. Let us try to further clarify the point.

We have mentioned that in our interpretation the Zande people are not *paraconsistent*, as some philosophers claim (we have seen the references before). We argue that when confronted with some consequence of their premises, such as that every man in a clan must be a witch, they refuse the conclusion; they avoid 'going there', that is, they do not follow the whole argument chain as probably we would do and stop the argument when it enables them to arrive to some point which interests them or which is enough to them: *this is enough*, they would claim; *we do not need more than this, let us change subject*. Can this reasoning be justified? I think it can.

A particular logic ℓ is a mechanism of inferences. In our Western way of reasoning, logic is normative; since once we have chosen a certain logic within a certain context, we become committed to its rules. If we disagree with such logic, we can try to change it, but this is, in general, a difficult and disputable step, since most times the considered theory has not an explicit underlying logic. But it can be done, at least in principle. Broadly speaking, logical inferences can be deductive, inductive (of several kinds) or even of some other nature; we leave open the possibility of different kinds of reasoning. Within a certain context, we reason from premises and follow the logical rules until the end, but the Zande case seems to present us with a different alternative, of course admitting that we can speak of Zande logic.[19] Anyway, we think that the Zande way of reasoning can also be guided by some 'logic', understanding this as a canon of inferences. Theirs would be a very strange logic (to us at least), a logic which could lead us to dogmatism or sectarianism since *we decide* when the argument should stop (but why should we refuse the same epithet to our preferred logic? Once we chose a 'logical way of reasoning', we are committed, in principle, to stop when the logic says we must). It seems that the Zande logic of 'not going there', if we can speak this way, should be amended by a meta-rule stating that *when some conclusion will be against your ways of reasoning or creates discomfort, simply ignore it*. This is a clear pragmatic aspect of logic, in the sense that the logic would be not independent of those who apply it (da Costa 1980, chap.1), so being not completely free from psychologism. We see, then, that the acceptance of different logics and pragmatic criteria for choosing brings back a problem which ever since Frege and Husserl's time philosophers think was ruled out from logic, namely, psychologism. This is of course something to be considered carefully, but of course not here.

Accepting a so wide dimension of logic, we can say that to reason within some context is to follow *some* logic, and different people can reason with different logics in the same context (the Azande on one side about witchery, ours on the another). Different contexts may require different logics and there is no a priori preferred logic out of pragmatic reasons. The normative character of logic, at least *our* Western logic (whatever it is), is not followed by the Azande, and neither by Wittgenstein.[20] To

Wittgenstein, as we have seen, arithmetics could be useful even if contradictory: simply do not go there; you can extract good consequences from the rest of it, and this would be enough. The same can be said about the standard scientist who is working on her theory, such as the Standard Model of Particle Physics or quantum mechanics, without the secure knowledge that it cannot lead us to contradictions. By the way, Roger Penrose, who won the Nobel Prize for physics in 2020, thinks that quantum mechanics is inconsistent since it assumes the 'unitary' evolution of the vector state by the Schrödinger equation and at the same time accepts that when a measurement is made, the state vector reduces to one of the eigenstates of the measured operator. The subject is subtle and still deserves attention.[21]

Finally, let us add that we think that this mentioned 'pragmatism' in choosing a particular logic or theory can be read also from a more metaphysical perspective, a perspective that guides our *Weltanschauung*. In our opinion, if some proposed logic or some scientific theory does not agree with the commonly shared worldview, it will need perhaps a lot of time to enter the mainstream, as the history of science exemplifies so well.

7. Conclusion

Of course, we cannot 'conclude' the discussion about inconsistencies and anomalies in science with definitive words. The subject is quite relevant and the philosophical analysis demands deep considerations. The fact is that we live with inconsistencies and when we realize them sometimes they become anomalies. Anomalies can sometimes be tolerated until a certain point, as the history of science has shown, and must be overcome, generally by modifying the paradigmatic theory or by introducing additional hypotheses that enable us to surpass the difficulties.

Anomalies turn into contradictions if and only if some logical analysis is given, which demands (at least) axiomatization, and we know that not every theory is put in an axiomatic form, yet this would be desirable for a better understanding of its basic principles and notions. So, dealing with anomalies and their considerations is a task for the philosopher, more than for the scientist.

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Notes

[1] Only after I had written the first version of this paper I took contact with the book *Inconsistency in Science* edited by Joke Meheus (2002) where there are chapters which give several references to things quite similar to those I have driven to in this paper. If I had known this book before, surely my text would be better.

[2]It is relevant to recall that in 1960 Abraham Robinson re-introduced infinitesimals (consistently) in Analysis by presenting Non-Standard Analysis (Robinson 1960) (see Bell 2008). It is also interesting to notice that Calculus in the old style is still useful for engineers and other applied scientists who are not occupied with questions about the foundations of their theories. They work with infinitesimals but do not push them to contradictions.

[3]Some good explanations about the paradoxes can be found in Russell 1976, §1, Fraenkel et al. 1973, §2, and Mendelson 1997.

[4]In my view, in some sense the electrons orbit regions **are** the electrons and their 'densities' give us an integer number of unities of charge, that of a sole electron, which we associate with a certain number of electrons. There are no **individual** electrons hidden in the orbital clouds, so it seems to be a mistake to say, as it is common, that the orbitals are the regions where there is more probability of finding the electrons, something that passes the idea that the electrons are something hidden within the clouds. We cannot catch one specific electron from there but just make the quantities increase or decrease of one unity say by ionization.

[5]We reinforce the claim that inconsistencies may exist unnoticed; anomalies come when the inconsistencies are noticed, but one can stay and leave them; but sometimes they are pursued or analysed and realised that contradictions appear.

[6]A situation where the presence of contradictions would not 'cause troubles' (that is, trivialization) is when the theory's underlying logic is paraconsistent. But in this case, one is faced with the (another) problem of explaining what negation means (Beziau 2002). If in a paraconsistent theory we may have both α and $\neg\alpha$ as theses, what does '$\neg$' mean? There are several distinct paraconsistent logics to be used for basing theories involving inconsistencies but not accepting the Explosion Rule. Inconsistent-adaptive logics (Batens 2000) are also logics that avoid explosion but in a different perspective. I treat them as paraconsistent as well.

[7]A Grand Unified Theory (GUT) would unify three of the four fundamental forces; the electromagnetic force and the weak force were unified in the Electroweak Theory, while the strong force is dealt with in Quantum Chromodynamics. Physicists suggest that they meet in a GUT whose gauge group would be termed $SU(5)$ (supposed to be $U(1) \times SU(2) \times SU(3)$), but (at least to my knowledge) the subject is not completely settled till now. See Glashow (1991); the entry Grand Unified Theory of Wikipedia is illustrative and updated. Roger Penrose says that quantum mechanics is inconsistent due to the two basic ways of evolution of the state vector (in Dirac's approach); the unitary evolution by Schrödinger's equation, and the collapse of the state vector after a measurement. There are several interviews with him on the web.

[8]Notwithstanding, it should be remarked that in 1960 Abraham Robinson re-introduced infinitesimals (consistently) in Analysis by presenting Non-Standard Analysis (Robinson 1960) (see Bell 2008)).

[9]Newton used to write things like $x + o$, but here we follow a more recent notation, usual in some old books on Calculus.

[10]The only possibility would be to assume some form of counterfactual conditional of the form 'If it were that A (a particle), then it would be that B (not a wave).' For the use of this kind of conditionals in quantum theory, see the entry by Lev Vaidman in Greenberg et al. (2009).

[11]According to them, quantum physics is the land of anonymity, where proper names make no sense since they to not play the role of rigid designators, as it would be if the involved entities were individuals.

[12]You can see in https://physics.aps.org/articles/v13/79.

[13]Lakatos, for instance, in criticising the axiomatic method, suggests that it does not involve 'creativity', since all theorems to be deduced are in a certain sense already implicit in the axioms.

[14]We also emphasize that Patrick Suppes sees heuristics also in the axiomatic method,

so its role would be not of just 'clean the house'; see Suppes (1983). So, the 'new results' can be in fact new. We agree with Suppes. Really, non-standard models of arithmetic, or of the real numbers could not be noticed without axiomatization pushed to its limits, that is, formalization.

[15]The name of this group of people is 'Azande', but 'Zande' (or 'zande') is also used to designate either the people or a particular person of belief and this is why we also find references to 'Zande logic', 'Zande beliefs', etc. See also da Costa et al. (1998), Jennings (1989), and the Introduction in Pritchard (1976).

[16]This means that there are no formulas β and $\neg\beta$ such that $\Delta \vdash \beta$ and $\Delta \vdash \neg\beta$, where $\vdash$ is the symbol of deduction of classical logic.

[17]Of course by supposing that the other formulas in Γ don't do that.

[18]For more details on this logic, see da Costa and Krause (2004) and the references therein.

[19]This depends on what we call 'logic', but here we are speaking very broadly.

[20]I am aware that to speak of Wittgenstein is a difficult task mainly because, as said Kielkopf, "Unfortunately, Wittgenstein did not *unpack* his insights" (Kielkopf 1970, p.3). I am not trying to say that he refused classical logic as known in his time, but just guessing that his conception of logic was not as something complete, actually done, but as something always being constructed (a 'strict finitism' for Kielkopf). In this sense, I think that his ideas are similar to Bourbaki's, yet Bourbaki accepted classical logic, while Wittgenstein was more concerned with its constructive part – of the same logic. Let me just add that according to Bourbaki, to do mathematics is to write symbols in a paper according to the classical rules (described in his Bourbaki 1968), so it is also in eternal construal, yet theorems like the Excluded Middle Law hold: one day when we have written enough symbols, the open problems will be unpacked.

[21]There are several interviews with Penrose on YouTube where he sustains such a belief.

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