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PHILOSOPHICAL INTERPRETATIONS MATTER

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Abstract. In recent years, there has been an increasing debate about some philosophical aspects of paraconsistent logics. The focus of this controversy has been on whether the notion of philosophical interpretation of a logic is separable or independent from the notion of application of a logic. The concept of application of a logic, or applied logic, comes from Priest (2005). Given a purely technical system, one can apply it to different domains, such as computer science, programming, formal linguistics, electric circuits, neural networks, control systems, reasoning, etc. On the other hand, Barrio (2018) and Barrio and Da Ré (2018) have argued that there is a third dimension of a logic that consists in its philosophical interpretation. This aspect is related to some intended meaning of the logical constants and the consequence relation of the pure system. In this sense, while Barrio and Da Ré have argued that, at least in some contexts, the concept of philosophical interpretation has an explanatory role, Arenhart (2022) has claimed that this concept can be reduced to the notion of application. In this article, we will show that the philosophical interpretation of a logic has a crucial role in the usual stances of logicians and philosophers, which can and must be distinguished from the application of the system. We will focus on paraconsistent logics as study cases for our points since the debate has developed almost entirely around these systems. However, most of our claims can be applied to any logical system. In fact, throughout the paper, we will also provide examples using non-paraconsistent logics.

Keywords: philosophical interpretations • paraconsistency • pure vs applied logics • substructural logics

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1. Introduction

In recent years, there has been an increasing debate about some philosophical aspects of paraconsistent logics. The focus of this controversy has been on whether the notion of *philosophical interpretation* of a logic is separable or independent from the notion of *application* of a logic. The concept of application of a logic, or applied logic, comes from Priest (2005). Given a purely technical system, one can apply it to different domains, such as computer science, programming, formal linguistics, electric circuits, neural networks, control systems, reasoning, etc. On the other hand, Barrio (2018) and Barrio and Da Ré (2018) have argued that there is a third dimension of a logic that consists in its philosophical interpretation. This aspect is related to some intended meaning of the logical constants and the consequence relation of the pure system. In this sense, while Barrio and Da Ré have argued that, at least in some contexts, the concept of philosophical interpretation has an explanatory role, Arenhart (2022) has claimed that this concept can be reduced to the notion of application. In this article, we will show that the philosophical interpretation of a logic has a crucial role in the usual stances of logicians and philosophers, which can and must be distinguished from the application of the system. We will focus on paraconsistent logics as study cases for our points since the debate has developed almost entirely around these systems. However, most of our claims can be applied to any logical system. In fact, throughout the paper, we will also provide examples using non-paraconsistent logics.

Let's start by introducing the idea of paraconsistent logic and its motivations. The basic intuition behind the adoption of a paraconsistent logic is the rejection of the classical principle known as (ECQ) *ex contradictione quodlibet* (from a contradiction everything follows), modernly known as Explosion. Thus, one can have different philosophical motivations for abandoning ECQ. Here, as some of us did in previous works, we will use a classification of these motivations due to Igor Urbas in (1990). Although the article is old (it was published almost 40 years ago), it is very topical. In his paper, Urbas classifies motivations towards paraconsistency into three classes:[1].

The first position is the dialetheism. In a nutshell, dialetheists consider that there are true contradictions. Usually, dialetheists use paraconsistent logics in order to deal with paradoxical phenomena. Among dialethists we could mention Asenjo (1966), Priest (1979), Priest (2006), Routley and Meyer (1976), Beall (2009), and Priest et al. (2023)among many others.

The second position is related to a pragmatic use of paraconsistent logics. These logicians use these systems in order to deal with inconsistent information, e.g. databases, collections of statements, beliefs, and scientific theories. However, they do not necessarily accept any deviant metaphysical position, such as dialetheists. In this position, we can mention the Brazilian tradition initiated by Da Costa (1974), and continued and developed among others by Carnielli, Coniglio, and Marcos in Carnielli et al. (2007), and Carnielli and Coniglio (2016).

The third position or main motivation is based on relevance. Roughly speaking, relevant logicians argue that in a valid argument, there must be some connection between the premises and the conclusion, i.e. the premises should be relevant to the conclusion. It's easy to observe that ECQ exhibits a highly irrelevant principle (see Mares (2024) for an introduction to relevant logics).

So, no matter the position or motivation, all the paraconsistent logicians and philosophers reject ECQ. However, as we mentioned before, there are many ways we can interpret and motivate the adoption of a pure system. What we will argue here is that philosophical interpretations play a crucial role in the choice of a pure system. Moreover, we will argue that philosophical interpretations are not reducible to applications of pure systems. In order to do that, the article will be structured as follows. In Section 2 we will introduce some logics that we will use to illustrate our arguments. In Section 3 we will present Barrio (2018)'s and Barrio & Da Ré (2018)'s arguments that show that a logic L may receive different philosophical interpretations. In Section 4, we present Arenhart's arguments that show that the notion of philosophical interpretation is an unnecessary addition to the already existing distinction between pure and applied logics. In Section 5, we respond to Arenhart's criticisms and argue that the notion of philosophical interpretations should be properly distinguished from applications and that the former plays a crucial role in our understanding of logical systems. In Section 6, we close the discussion with general remarks.

2. The logics

In this section, we will introduce a family of logics, which we will use throughout the paper to illustrate our claims. Some of them are paraconsistent logics. Technically speaking, the following is the formal incarnation of the principle of ECQ:

Definition 2.1. A logic L is paraconsistent if and only if $\varphi, \neg\varphi \nvDash_{\mathsf{L}} \psi$, for some $\varphi, \psi \in For(\mathscr{L})$.[2]

In this article, we will work with paraconsistent systems based on the Strong Kleene schema:

	¬
1	**0**
i	**i**
0	**1**

∧	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**0**

∨	**1**	**i**	**0**
1	**1**	**1**	**1**
i	**1**	**i**	**i**
0	**1**	**i**	**0**

Figure 1: Strong Kleene truth-tables

Let a valuation be a function v from formulae to the set $\{\mathbf{1}, \mathbf{i}, \mathbf{0}\}$. Following Chemla et al. (2017) we can define five logics using these tables:

($\mathsf{K_3}$) $\Gamma \vDash_{\mathsf{K_3}} \Delta$ if and only if for every valuation v,
if $v(\varphi) = \mathbf{1}$ for every $\varphi \in \Gamma$, then $v(\psi) = \mathbf{1}$ for some $\psi \in \Delta$.

(LP) $\Gamma \vDash_{\mathsf{LP}} \Delta$ if and only if for every valuation v,
if $v(\varphi) \in \{\mathbf{1}, \mathbf{i}\}$ for every $\varphi \in \Gamma$, then $v(\psi) \in \{\mathbf{1}, \mathbf{i}\}$ for some $\psi \in \Delta$.

(ST) $\Gamma \vDash_{\mathsf{ST}} \Delta$ if and only if for every valuation v,
if $v(\varphi) = \mathbf{1}$ for every $\varphi \in \Gamma$, then $v(\psi) \in \{\mathbf{1}, \mathbf{i}\}$ for some $\psi \in \Delta$.

(TS) $\Gamma \vDash_{\mathsf{TS}} \Delta$ if and only if for every valuation v,
if $v(\varphi) \in \{\mathbf{1}, \mathbf{i}\}$ for every $\varphi \in \Gamma$, then $v(\psi) = 1$ for some $\psi \in \Delta$.

From these four systems, two of them LP (Priest 1979) and TS (Cobreros et al. 2012) are paraconsistent logics. For example, the valuation $v(\varphi) = \mathbf{i}, v(\psi) = \mathbf{0}$ is a counterexample to $\varphi, \neg\varphi \vDash \psi$ in LP. On the other hand, TS is a paraconsistent logic since is an empty logic (every inference has a counterexample).

Notice that ST is not a paraconsistent system since ST coincides with classical logic in the valid arguments. However, ST differs from CL in the valid metainferences. Actually, among others, the following two are not valid in ST:

$$\text{Cut}\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad\qquad \text{Meta Explosion}\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi \qquad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta}$$

In virtue of the failure of these metainferences, Barrio, Pailos, and Szmuc (in Barrio et al. 2018), and also Da Ré, Rubin, and Teijeiro (in Da Ré et al. 2022) have argued that ST as a metainferential logic is also metainferentially paraconsistent.[3]

What is common to these logics is that all of them have been applied to contexts where some semantical vocabulary is involved. So, we can safely claim that all of these logics share this application as the main one. However, all of them have very different philosophical interpretations. In particular, the intermediate value **i** has been interpreted as a gap, a glut, or as being meaningless, or off-topic, providing with each of these interpretations also different conceptual accounts of the notion of logical consequence and the connectives[4]. Also, although the main application of these logics is to paradoxes, some of them have other applications, under the same interpretation. And even some of them for the same application have received different philosophical interpretations.

In a nutshell, a logic has three conceptually distinguishable aspects: the pure system, its applications, and the (philosophical) interpretations that receives. These three faces are distinct, can be distinguished and, at least in some context, each of them contributes to the whole understanding of what a logic is. In the following sections, we will argue for these things in detail.

3. Logics have multiple interpretations

The philosophical debate about logics and their respective interpretations is a topic of recent discussion in the literature, and the case of paraconsistent logic has received considerable attention in this debate. As we mentioned, paraconsistent logics were proposed to deal with inconsistencies that appear in our reasoning, and the interpretation of these inconsistencies is an exciting object of discussion. Many philosophers and logicians have interpreted paraconsistent systems in alethic terms and postulated the existence of *dialetheias* (Priest et al., 2023; Priest, 1979, 2006), which are sentences φ that are true and false. On the other hand, many other philosophers and logicians have interpreted these logics in informational terms, such as in Carnielli et al., 2004; Carnielli and Rodrigues, 2012, 2015a; Belnap, 2019. In this latter perspective, contradictions result from the flow of information, from our scientific theories, and so do not reflect anything in reality. Although these two positions are widespread and can be considered well-established ones, as argued in Barrio and Da Ré (2018), we think that there is not a necessary connection between a paraconsistent logic and some particular interpretation.

Regarding the literature about the interpretation of paraconsistent logics, we find arguments defending that contradictions are not ontological. That is, all inconsistencies arise in our theoretical apparatus or in our databases. In this sense, paraconsistent logics should be exclusively interpreted from an epistemic perspective. However, there are criticisms against these interpretations of these systems. As we mentioned, Barrio (2018) and Barrio and Da Ré (2018) argue that nothing in the formal systems forces us to interpret them in some particular way. Barrio and Da Ré argue that neither *dialetheism*, which asserts the existence of dialetheias, nor the epistemic can be considered as the canonical interpretation of paraconsistent logics. In a more general perspective, Barrio (2018) argues against the thesis that logics has a canonical interpretation. In what follows, we present their objections.

According to Priest (2005), pure logics are mathematical structures, defined by means of proof-systems or model-theoretically, endowed with a consequence relation. At this level, the logics only establishes what logically follows from what. On the other hand, they can be applied to diverse domains, such as electric circuits, programming, reasoning, and so on. There are many pure logics and no rivalry between them. The dispute between them begins only in the field of applications. That is, only in the field of applications do we discuss which logic is the most appropriate for a given context.

For Priest, the canonical application of logics is in natural language reasoning. So, according to Priest, only one logic is the right one if natural language reasoning is the unique canonical application.

So far, we have pure logic and applied logic. The former denotes the system in the technical sense, whereas the latter refers to the concrete domain in which we apply the logic. However, there is a third aspect: the interpretation of the logic. Although Barrio and Da Ré do not explicitly define the notion of philosophical interpretation, we can understand this notion as an analysis (a conceptual description) of the logical vocabulary of constants of L in terms of an informal notion, and the notion of validity of the logic. For example, before Kripke and others' works, modal logics were well-known logics from a technical point of view, and they also had applications. However, the idea of interpreting the points of the models as possible worlds, the relations as accessibility relations between worlds, and the idea of truth preservation in every world have changed the way we look at modal logics. In some sense, we can say, we have a different access to the understanding of these systems; the interpretation seems to be playing some role in our relation with the logics.

Moreover, Barrio and Da Ré argue that, given a pure or an applied logic L, there is no formal constraint in the system that forces it to be philosophically interpreted in a unique way. According to them, a philosophical interpretation is employed to give an additional understanding of a pure logic, giving meaning to the logical constants of L. Even in the case that L is being applied to reasoning, there is no formal constraint in L that prevents different philosophical interpretations. In the case of paraconsistent logics, they argue that paraconsistency is a property of those systems whose negation does not validate the explosion rule, but this by itself is not sufficient to force any interpretation. As they show, for example, it is possible to adopt a paraconsistent logic without being a dialetheist, and it is even possible to be a dialetheist without adopting a paraconsistent logic[5].

As one can observe, the notion of philosophical interpretation is quite general, having to do with the attribution of meaning to the logical constants of a given pure logic. Given a philosophical interpretation i of a pure logic L, the formal consequence relation of L is informally read in terms of the informal consequence relation induced by i. In this sense, a philosophical interpretation is a structure-preserving mapping from pure logic to a conceptual reading of the more important elements included in the theory.

Giving a philosophical interpretation i of pure logic L allows us to accurately discuss intuitions about a conceptual network involving the notion i. For example, the modal operator $\Box$ of S5 captures the concept of *logical truth* (Field, 1991; Burgess, 1999). So, in this case, the axioms and the theorems of S5 express the general properties of these informal concepts. Thus, we are allowed to say that S5 gives us an accurate understanding of the notion of logical truth. Modal logics are interesting formal tools because they allow several philosophical interpretations. For instance, in epistemic logics, we find different modal logics whose vocabulary is interpreted in epistemic terms. Different epistemic modal logics capture different aspects of the informal notion of knowability, and this allows us to compare the advantages and disadvantages of adopting one or another pure logic.

Under this understanding of philosophical interpretation, this notion is close to what Benito-Monsalvo (2022) calls *external interpretation*, which is a notion "that has to do with the philosophical justification of a logic; i.e., with an explanation of why our meaning specification of the logical vocabulary is such that it makes some inferences valid and not others" (Benito-Monsalvo 2022, p.537). Both notions are quite general, also comprehending informal interpretations that are not necessarily philosophical. The fact that we call philosophical a given interpretation has to do with the philosophical attitude of justifying the meaning of the logical constants of a pure logic. Here we will use the notion of philosophical interpretation, in order to maintain the terminology adopted in Barrio and Da Ré (2018)'s and Barrio (2018)'s works.

So, the notion of philosophical interpretation is clearly distinguished from formal interpretation. Indeed, this latter notion has to do with model theoretical apparatus that interprets the language of L and defines the model-theoretical consequence relation $\models_{\mathsf{L}}$. On the other hand, as we argued before, philosophical interpretations are informal. They have to do with the analysis of the informal meaning of the logical constants and the notion of logical consequence. In this sense, they can capture a preexisting deductive practice underlying the informal meaning of the main logical expressions of a formal theory. For example, classical logic might be viewed as formalizing our deductive practice involving the preservation of truth and relevance logics as capturing the analytical entertainment.

> What seems clear to us is that the informal concept of consequence at work in natural mathematical languages is often plainly semantic, and moreover model-theoretic. That when the mathematician draws inferences in natural language, s/he imagines a situation in which the hypothesis is true—i.e. one has a model for the hypothesis in view—then s/he argues that the conclusion must hold in that model. (Kennedy and Väänänen, 2017, p.6)

So, what Kennedy and Väänänen argue is that classical first-order logic formalizes mathematical reasoning by preservation of truth. A similar thing can be said concerning intuitionistic logic and its formalization of the preservation of constructive provability because Brouwer-Heyting-Kolmogorov interpretation (BHK-interpretation) is intended to be an interpretation of the logical constants of intuitionistic logic IL (van Dalen 1986). As Kripke (2022) observes, it is in virtue of the BHK-interpretation that intuitionistic logic was proposed in order to formalize the general principles of this interpretation:

> Intuitionists, I think, are not really proposing to modify the laws of logic with the classical connectives. They propose new connectives, more appropriate according to them for mathematics, which satisfy different laws. One sees, by virtue of the interpretation that they gave to their connectives and quantifiers, that a different set of laws is satisfied by these distinct concepts. (Kripke 2022, p.21)

So, as the above passage highlights, BHK-interpretation gives informal meaning to the logical constants of IL. We say informal because the concept of methods of construction underlying BHK-interpretation is unspecified (Iemhoff et al. 2001). In this sense, this logic is a formalization of the deductive practice involving the preservation of constructibility. Notice that this is quite different from just adopting a logic as a mere formalism (as quantum logic in the renowned case of Putnam) and using it for some particular application. Here, BHK-interpretation besides its application provides a philosophical interpretation of the logic.

Going back to the context of paraconsistent logics, Barrio (2018) also argues against Carnielli and Rodrigues' approach to paraconsistency (Carnielli and Rodrigues, 2012, 2015b, 2019a,c,b). Roughly speaking, departing from Priest's interpretation of paraconsistent logics, captured by the logic LP (Asenjo 1966; Priest 1979, 2006), Carnielli and Rodrigues claim that the notion of evidence is well-suited for interpreting paraconsistent logics. The authors understand evidence for a statement *A* as *reasons for accepting/believing A*. They argue that the *basic logic of evidence* (BLE) and the *logic of evidence and truth* ($\mathsf{LET_J}$), which extends BLE with a classicality operator $\circ$, formalize the notion of preservation of evidence. Although they recognize that both systems can receive other interpretations than their proposed one, their paper (Carnielli and Rodrigues 2019c) generated an interesting discussion in the literature. In the same spirit as in (Barrio and Da Ré 2018), Barrio argues that there is nothing in the formal systems BLE and $\mathsf{LET_J}$ that forces an epistemological interpretation in terms of evidence. Indeed, Barrio argues that it is possible to interpret both systems in alethic terms, allowing both dialetheias and truth-values gaps. Such a thing is possible since BLE is equivalent to Nelson's logic N4, which can be characterized by means of a four-valued possible worlds semantics. These values can be interpreted as: *just true*, *just false*, *both true and false*, *neither true nor false*. Both the alethic and epistemic interpretations are equally legitimate from the philosophical point of view. In this case, none of these interpretations should be taken as canonical for these two logics.

Both Barrio and Barrio and Da Ré's objections also apply to other logics. As formal languages, logics are not able to tell us what are their philosophical interpretations. There are different, but equally suitable, philosophical interpretations. In a more general perspective, we could also mention, following Bezerra and Venturi (2021), that the consequence relation of a logical system cannot capture a unique informal notion of logical validity. That is the formal notions of validity of a particular logic L capture different informal notions of validity. One could say that each informal notion of validity carries its own philosophical interpretation from which the logical vocabulary of L will be interpreted.[6]

So far we have presented and discussed the main positions of Barrio and Da Ré regarding the importance of philosophical interpretations and their differences with pure logics. In the next section, we will introduce some objections that this position has received, and later on, we will try to respond to such criticism with new arguments.

4. Are Interpretations determined by applications?

Although Barrio and Da Ré's objections successfully show that paraconsistent logics are not tied to a specific interpretation, they also were objects of criticism. Arenhart (2022) argues that the notion of philosophical interpretation is an unnecessary addition to the well-known distinction between pure and applied logic. According to his position, once one logic is selected to be applied to a specific context, the notion of philosophical interpretations plays no significant role. In what follows, we present his arguments.

In the case of paraconsistent logics, Arenhart argues as follows: dialetheism is not an interpretation of paraconsistent logics. As defined by Priest et al. (2023), dialetheism is an ontological view about the nature of truth, which says that there are contradictory pairs A and $\neg A$ that are both true. So, Arenhart argues that seeing dialetheism as an interpretation of paraconsistent logics is a misconception about this ontological thesis. For him, these logics are applied to formalize theoretical contexts where dialetheism is the case because these contexts require a logic that dispenses the law of explosion. The following passage testifies Arenhart's argument:

> The problem with this account is that dialetheism, understood as a philosophical interpretation, is not doing any work here. Seeing this is a matter of putting the issue in the appropriate perspective. To begin with, this is not how dialetheism should be seen, and that is not even how Priest sees it, it seems to us (see also Arenhart 2020, p.11548). That is, dialetheism is not a thesis about how paraconsistent logics are to be read or understood, but it "is a quite general metaphysical/semantic view about truth and negation" (...) That is, using dialetheism to interpret LP is to see things from the wrong perspective; one confuses the priorities. Paraconsistent logics are applied to the study of reasoning in circumstances where dialetheism is true (if it is true at all, of course), and Priest subscribes to a very specific paraconsistent logic because the field of application with which he is concerned, reasoning, is deeply impacted if dialetheism is correct. (Arenhart 2022, p.448)

The same happens with the evidence approach to paraconsistency. As he argues in (Arenhart 2020, 2022), what Carnielli and Rodrigues are really doing is applying a paraconsistent logic that models reasoning by evidence.[7] That is, both logics BLE and $\mathsf{LET_J}$ should be taken to formalize the notion of preservation of evidence. In both cases, dialetheist and epistemological approaches, Arenhart says that the notion of philosophical interpretation plays no significant role. The only relevant distinction is the pair pure *versus* applied logic. In the case of dialetheist, one could take the logic LP to model dialetheic reasoning. In the case of the epistemological approach, one could adopt the systems BLE and $\mathsf{LET_J}$ to formalize the idea of preservation of evidence.

Now, the reformulation of Barrio and Barrio and Da Ré's objections go as follows: Barrio argues that BLE and $\mathsf{LET_J}$ can receive an alethic interpretation, given that it can be interpreted by means of the aforementioned four-valued possible worlds semantics that characterizes Nelson's logic N4. On the other hand, one could argue that these four truth values can be applied both to the alethic context and to model reasoning involving the notion of information. Thus, the same formal semantics has two distinct applications. In both cases, the notion of philosophical interpretation does not play a fundamental role, in such a way that it can be dispensed.

The same could be said with respect to LP. Although it may be defended that it can be applied to formalize dialetheist basic intuitions, this logic can be applied to other notions, such as the concept of *truth in a desambiguation* (Pinter 1980; Lewis 1982). Moreover, LP can be characterized by distinct formal semantics, such as its original matrix semantics, bivaluations (Bezerra 2020), supervaluation semantics (Priest 2008), non-deterministic matrices (Rosenblatt 2015) and so on. Each of these semantics was proposed to be applied to distinct contexts, which shows that nothing in LP says that it can only be applied to a unique theoretical field.

From a more general perspective, the situation is quite similar. A particular logical system L may be applied to diverse contexts in principle. In each application, the notion of philosophical interpretation does not play a significant role, which means that it can be dispensed to show that a formal system is silent with respect to its possible applications. Of course, this does not mean that L can be applied to every context, or at least it is not implied from the above claims.

In the following section, we will discuss in detail all these arguments and provide examples in order to show that, actually, philosophical interpretations matter.

5. On why philosophical interpretations matter

In this section, we will argue for one main point: in spite of Arenhart's criticisms philosophical interpretations should be distinguished from applications since they play a crucial role in our understanding of a logical system.

As we saw in Section 4, Arenhart argues that the distinction between pure and applied logics is enough to show that logics can be applied to distinct contexts and that the notion of philosophical interpretation plays no significant role besides this distinction. However, we think that the notion of philosophical interpretation still has an explanatory value and that it should be distinguished from the notion of application of a logic. In this line, we fully agree with the following words (Antunes and Szmuc, 2022):

> (...) one of the key notions that is crucial for achieving a comprehensive understanding of the role of logic is that of interpretation. For if a logical system is to have any significance beyond that of a mere mathematical structure, then it appears that it must be possible to endow it with some sort of interpretation (and this holds true even in those cases in which only a certain technical application is envisaged) (Antunes and Szmuc 2022, p.177)

In order to show that the philosophical interpretation of a logic must be distinguished from its applications we will present three main arguments. First, we will claim that the discussions around whether there are one or more correct logics when applied to a certain domain presuppose the notion of philosophical interpretation. Secondly, we will argue that collapsing interpretations and applications would obscure the explanatory role of each concept. Lastly, we will show that for one application there can be more than one interpretation. We will illustrate this point using as study cases the paraconsistent substructural logics ST and TS and their application to semantic paradoxes.

So, in what follows, we will argue that the very discussion about the correction of a logic for an application assumes the notion of philosophical interpretation. Pure logical theories are interesting, of course. At this level, general logical features can be analyzed in the abstract, compared with other systems and systematically studied very important metalogical aspects such as the correctness and completeness of a test system, the existence of algorithms to carry out these tests, etc. However, pure logical theories, even applied to a specific domain, are interesting, but without a conceptual interpretation of their main features, without a conceptual analysis that allows us to understand what kind of consequence relationship, and what language features model their connectives, there is no possibility of determining which logics are correct and which are not, which ones we should discard or adopt as explanations of our inferential practices. This doesn't mean that one should be aware of the interpretation of a logic when one is applying it. For instance, consider a student of Physics who applies classical logic to electric circuits. It may be the case that she misses, or is not interested in, some conceptual aspects of classical logic. However, she can use the logic and formally explain how it works. However, once the student wants to explain why classical logic is the correct logic as an explanation of her inferential practices, she must provide some interpretation beyond the technicalities, i.e. a philosophical interpretation.[8]

Let's illustrate what we have claimed in the last paragraph with some well-known examples. The adoption of classical logic to capture the reasoning already presupposes that the vocabulary of this logic can be adequately interpreted in terms of truth preservation and consequence relation in terms of necessary truth preservation. The same applies to intuitionistic logic with respect to the preservation of the constructibility of proofs and to relevant logic with respect to the preservation of content. It would be incorrect to apply classical logic to constructive reasoning since there are classical principles that are invalid according to the constructive interpretation of logical consequence. That is, the application of each of these logics to reasoning presupposes a philosophical interpretation that must be adequate to their formal vocabulary.

As we have seen, the same pure logic can receive different philosophical interpretations, even in the same application. And the revision of a logic can be a matter that depends on how to conceptually interpret that same formal structure. Just for emphasis, our main point here is that from the fact that there are multiple philosophical interpretations for the same pure logic, it does not follow that the concept of interpretation is unimportant, nor that it should be collapsed with that of application.[9]

Moreover, we think that there is another main reason to distinguish interpretations from applications. Namely, they play different explanatory functions when we theorize about logics. The fact that as Barrio (2018) and Barrio and Da Ré (2018) claim there is no a 1-1 correlation between pure logics and applied logics, it doesn't mean (as Arenhart seems to have argued) that interpretations are subsumed into applications. In fact, these two concepts play different roles. The interpretation, as we understand it, plays the role of providing an analysis of the meaning of the connectives and the consequence relation. On the other hand, the application refers to the act of using a system in a certain domain. For instance, one could perfectly apply a logic to an electronic process without being committed to any particular interpretation or conceptual analysis of the logical vocabulary involved. Although we admit that the distinction between both concepts is blurred in some limit cases, we think that making them coextensive just deprives us of one level of analysis.

Lastly, a pure formal structure with a particular application can receive many philosophical interpretations that provide an intuitive meaning of the connectives and the notion of the consequence relation. Of course, the interpretation will interact with the application, since the logical structure of the theory is being interpreted in some particular way. In order to illustrate that for one application there can be more than one interpretation, and thus both should be properly distinguished, we will consider the case of non-classical (paraconsistent) solutions to semantic paradoxes. The main reason is that some of these logics while being applied to the same domain, i.e. semantic paradoxes, can nonetheless receive different philosophical interpretations.

Semantic paradoxes are conundra involving semantic predicates, most notably the truth predicate. For instance, the Liar paradox, which consists of a theory containing a sentence, let's call it λ:

$$(\lambda)\ \text{This sentence is false}$$

If λ is true then what it says is the case, and therefore it is not true. On the other hand, if it is not true then what it says is not the case and therefore is true. In any case, we are in an impossible situation. As the previous example shows, if one accepts some intuitive principles regimenting the behavior of the truth predicate, we should abandon some classically valid principle.

As the literature testifies, there are many solutions to semantic paradoxes. Among these solutions, the substructural solutions became popular because they are able to block strengthened versions of these paradoxes by blocking some properties of the consequence relation. Here we will focus on the (paraconsistent) substructural solutions introduced in Section 1, Let's start with TS. As shown before, this logic is nonreflexive. As proved by (French 2016; Murzi and Rossi 2022), TS blocks the semantic paradoxes of truth and validity due to its non-reflexivity. So, one could want to apply this logic in order to deal with semantic vocabulary.

However, there is not a unique interpretation of this system. As Calderón and Pailos (2022) argue, the consequence relation of TS can receive an alethic interpretation, where the truth value **1** stands for *truth*, **0** for *falsity* and **i** for *indeterminacy*. In this way, the consequence relation of this system is interpreted as the preservation of non-falsity of the premises to the truth of the conclusions. On the other hand, another possible interpretation of TS is the bilateralist interpretation (Malinowski 2007). According to this reading, $\Gamma \models_{\mathsf{TS}} \Delta$ is read as "if we do not reject all the members of Γ, then we accept some member of Δ." So for the same application, we have more than one interpretation. We think that choosing one of them has a major impact on the way we understand the semantical phenomenon we are dealing with.

Although TS can be non-trivially extended with the semantic predicates of truth and validity, its application is quite restricted. In TS the Liar sentence λ cannot be proved, because no sentence is valid in it. As mentioned before, TS does not have either valid inferences or tautologies. The presence of the value **i**, along with the definition of the consequence relation of TS, makes every inference invalid. In particular, the TS solution to paradoxes consists in disproving both:

$$\not\models_{\mathsf{TS}} \lambda \text{ and } \not\models_{\mathsf{TS}} \neg\lambda$$

Thus, TS blocks the semantic paradoxes at the cost of counterexemplifying every sentence, even the non-paradoxical ones (as the sentence "*1+1=2* is true"). So, although we can say that TS has been interpreted in the way we mentioned before, and there is nothing wrong with the pure system either, however, there is something unsatisfactory with the TS solution to paradoxes. How can a solution to paradoxes consist of an empty logic? Here it becomes evident the distinction between application and interpretation.

Thus, this flaw of TS seems to be related to the way it solves the paradoxes and not to the philosophical interpretation of the system. So, let's take a stronger system: ST. This logic coincides with classical logic in the inferences it validates. However, as shown in Ripley (2012), it can be conservatively extended with a transparent truth predicate. In other words, the logic can deal with semantic paradoxes. And since this is a system that coincides with classical logic, one could argue that a possible application can be to formalize human reasoning. Moreover, as shown by Fjellstad (2016), it is possible to define Peano Arithmetic over this logic and extend it with a transparent truth predicate. In the resulting theory, the liar sentence will be provable as well as disprovable in the sense that we will have a derivation of both:

$$\vDash_{\mathsf{ST}} \lambda \text{ and } \vDash_{\mathsf{ST}} \neg\lambda$$

but since in ST meta- Explosion is invalid, i.e. the following instance:

$$\text{Meta Explosion}\ \frac{\Rightarrow ST\neg\lambda \qquad \Rightarrow ST\lambda}{\Rightarrow ST}$$

is invalid and therefore ST blocks the paradox in a paraconsistent way, i.e. by invalidating a version of Explosion. Notice, also, that it is just a case of the more general failure of the rule of cut (transitivity) which also blocks the derivation from $\Rightarrow \lambda$ and $\lambda \Rightarrow$ to absurdity. Going back to the interpretations, again, this system has different philosophical interpretations, for the same application, i.e. semantical paradoxes. For instance, we can give an alethic interpretation or a bilateralist one (see Cobreros et al. 2012; Ripley 2013). Even more recently in Priest (2023) discusses whether ST has even a dialetheic interpretation.[10]

Thus, even in the case of ST which can be seen as a more suitable formalism for applying to solve or deal with semantic paradoxes, however, there is no consensus on how the logic must be interpreted. Not only that, there is no dispute about it, since there is nothing neither in the pure system or in the application that can solve this problem. The interpretation of a logical system is a matter of philosophical stances that are in principle separated from applications.

Another very instructive example of the difference between application and philosophical interpretations is the case of non-contractive theories of truth. It's not important to go into the details of these proposals, but just to show how the non-contractivists discuss their theories, e.g. Rosenblatt claims (2021, p.2697):

> What we are after, then, is a philosophical explanation of the failure of structural contraction. The explanation should indicate why it is that contraction fails for paradoxical sentences and why it holds, if it does, for non-paradoxical sentences. The main task of the paper is to suggest an explanation that arguably achieves this.

In this quote, it becomes clear that there are two separable and distinct spaces. On the one hand, we have the application of some pure logic to solve semantic paradoxes. On the other hand, we read Rosenblatt asking (and providing) for a philosophical explanation, an interpretation, in other words, some informal reading of the failure of contraction, which is a property of the logical consequence relation. So again, we see that these two aspects can be distinguished, and not only that, the philosophical interpretations are essential to the project of making sense and understanding the application of a logic, and also comparing it to other solutions.

On the other hand, many non-classical theories of truth fail to adequately formalize our reasoning involving the truth predicate. Although the vocabulary of these theories receives an alethic interpretation, their application is severely limited to the solutions of semantic paradoxes due to the deductive weakness of their corresponding logical theories. An interesting example is given by Picollo (2020), who shows that truth theories based on the logic LP°, obtained by extending LP with the classicality operator $\circ$, fail in validating instances of the arithmetic induction axiom. That is the deductive weakness of a logical theory directly affects our mathematical theories, which restricts the application of these theories considerably. The severity of some of these solutions also affects our reasoning about truth.

So far we have been focused on the distinction between philosophical interpretations and applications in the context of semantic paradoxes. However, although this sole example is enough to show that these concepts can be distinguished, this should not be taken as something peculiar of this application. As another example of paraconsistent logics where we have an application and different interpretations for the same pure logic, we can consider the case of relevant logics. The application of these logics is conditionality, i.e. conditional statements of the form *If A, then B* in natural language. So, relevant logics are usually equipped with some extensional connectives and a primitive (intensional) conditional which intends to represent, in the logic, *the* conditional. The formal semantics of the (pure) relevant logics is usually described in terms of a Routley–Meyer semantics, where the notion of satisfaction of a conditional in a model is given by a ternary relation (for a general introduction to these concepts, see Mares 2024). However, as the authors show in Beall et al. (2012), mere pure logic is not sufficient for giving an account of conditionality. One also expects some additional elaboration on the conceptual interpretation of the formal apparatus with some analysis of the interpretation of the logic in the particular application. Thus, in order to fill the gap, the authors provide three different interpretations of the pure systems for the same application. In other words, we have another case of one pure logic, one application, but different philosophical intepretations.

These examples show that the notions of philosophical interpretation and applications do not collapse. Of course, as Benito-Monsalvo (2022) observes, philosophical interpretations are generally done in light of a specific application of the logical system. However, the examples of the non-classical solutions of semantic paradoxes show that the applications of some logics are considerably narrower than their interpretation.

As the following passage shows, Rodrigues and Carnielli also argue that the notions of philosophical interpretation and applications do not necessarily coincide.

> As we understand it, a philosophical interpretation of a logic is the intended meaning attributed to its expressions motivated by, or connected to, philosophical concepts. In order to be an applied logic, a formal system has to have intended meanings attributed to its expressions. These intended meanings, at first sight, may come without a philosophical interpretation, but it is reasonable to suppose that any applied logic is amenable to a conceptual — and so philosophical — discussion, given that in formalizing a domain a logic says things about that domain. Depending on how we understand the concept of an applied logic, it may be that a formal system has a philosophical interpretation but no application, the latter understood in a strict sense. (Rodrigues and Carnielli 2022, p.326)

So, Rodrigues and Carnielli argue that the notion of application assumes a philosophical interpretation. That is, the philosophical interpretation is seen as a step prior to the application of a logical system since it is the philosophical interpretation that gives informal meaning to the constants and the consequence relation. On the other hand, Arenhart (2022) argues that philosophical interpretation, understood in these terms, would be a formal semantics with an application requiring that concepts in formal semantics have philosophical appeal. He also argues that this multiplies the category of interpretations because an engineer can create the category of 'electric engineering interpretations' to distinguish his interpretation from the philosophical ones.

Nevertheless, we don't agree with Arenhart's objection to the above quotation. We think that it could be the case that the notion of philosophical interpretation is understood in quite general terms, having to do with the attribution of the meaning of logical constants as well as analyzing, in terms of this informal interpretation, what inferences are valid and what are not. So, this multiplication of categories of interpretations, such as philosophical interpretation, electric engineering interpretation, and so on, is not needed. Suppose an engineer who deals with electrical circuits says that his informal interpretation of the classical propositional logic is not philosophical. In that case, this is not a problem, provided that his interpretation is accompanied by an attribution of the meaning of the logical constants of this preferred logic and an explanation that legitimates the validity of the valid inferences and rejects the non-valid ones.

Before closing this section, it is important to reinforce that our arguments are not restricted to the paraconsistent case. Let us consider again the case of IL. We know that it is usually said that this logic captures the inferential meaning of the propositional connectives (Garson 2013). On the other hand, Gödel (1986) proved that IL can be embedded into the modal logic $\mathsf{S4}$, whose connective $\Box$ can be interpreted as *informally provable*, that is, provable by any correct means. As Gödel himself observes, $\mathsf{S4}$ validates principles such as $\Box\varphi \rightarrow \varphi$, which are incompatible with provability in the constructive sense.[11] It is also well-known that IL can be translatable into other modal systems, namely $\mathsf{S4Grz}$ (Grzegorczyk 1967) and KGL (Goldblatt 1978). In $\mathsf{S4Grz}$, $\Box\varphi$ is interpreted as "φ is provable and true", where as it is interpreted as "φ is provable in PA" (Solovay 1976). So, even if it is applied to constructible reasoning as well as to capture the inferential meaning of the connectives, it is compatible with distinct interpretations due to its translations into modal systems.

Keeping philosophical interpretation and application as separate notions allows us to focus on the adequacy conditions for interpreting a pure logical system. As we argued, there are logics whose philosophical interpretations are adequate, but their applications are somewhat problematic. So, the conditions for interpreting them and the conditions for applying them are different. The fact that both notions do not collapse shows that both tasks, interpreting a logical system and applying it, are distinct conceptual activities. In this respect, Tajer and Fiore (2022) argue as follows:

> [In this section,] we have presented three senses in which a pure logic may be interpreted: you can determine the application of the logic (thus specifying the intended reading of the logical constants), you can interpret the elements of the metatheory (such as the semantic values and the property of validity), and you can characterize the relevant notions philosophically associated with validity (e.g. 'truth', 'assertability', 'preservation', etc.). (Tajer and Fiore 2022, p.219)

And they add:

> Lastly, we note that the question of whether the notion of 'interpretation' should or should not be used is to some extent a mere terminological issue. (Tajer and Fiore 2022, p.220)

In the context of Fiore and Tajer's article, they don't need more than this minimal position to achieve their conclusions. However, we should emphasize that the difference between applications and philosophical interpretations is not a terminological issue. Reducing interpretations to applications obscures the actual practice of logicians and philosophers. We usually look for philosophical interpretations in order to increase our understanding of a system, but also because they play an explanatory role in the application of the logic. Of course, we are not claiming that all the aspects of a logic are independent meaning that makes any sense to interpret a logic that is not applicable or that lacks a pure presentation. These aspects are usually interlaced in our practices. What we are claiming is just that these dimensions are in principle distinct and play different roles. So collapsing these aspects deprives us of a useful and interesting conceptual dimension of analysis.

6. Conclusion

In this article, we have discussed the relation between the philosophical interpretation of a logic and its application. Building on previous work by Barrio (2018) and Barrio and Da Ré (2018) we have argued that philosophical interpretations have a conceptual role. Also, we have discussed the arguments provided by Arenhart (2022) who claimed that philosophical interpretations can be reduced to applications. In that sense, we have provided some examples based on logics applied to semantic paradoxes but having many distinct philosophical interpretations. These interpretations shed light on how we are explaining the semantic phenomenon to which we apply our logics. In a nutshell, we think that philosophical interpretations matter.

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Notes

[1]We are not aiming to be exhaustive about this topic. For a comprehensive presentation of the motivations and history of paraconsistent logics, see e.g. Priest et al. 2018, Carnielli et al. 2007, Carnielli and Coniglio 2016 or Gomes and D'Ottaviano 2017.

[2]There is another definition of paraconsistency related with the failure of the following principle: $\varphi \wedge \neg\varphi \nvDash_{\mathsf{L}} \psi$, for some $\varphi, \psi \in For(\mathscr{L})$. For the logics we will present in this paper, both principles are equivalent.

[3]Barrio and Pailos (2021) argues that a logic is not only a set of inferences.

[4]As we will argue, to provide an interpretation of the semantics values (an *applied semantics* in Copeland (1983) terms) is not enough to give a complete philosophical interpretation. In other words, the interpretation of the semantics values is just a part of a philosophical interpretation of a logic. We thank an anonymous reviewer for pointing this out to us.

[5]The authors show that it is possible to provide a dialetheist interpretation to the logic ST, although under the usual definition of paraconsistency, i.e. failure of Explosion, ST is not a paraconsistent logic! This point is also made by Ripley in 2013, p.155. However, one could accommodate the definition of paraconsistency in order to convert ST into paraconsistent logic. This is the route exlored in Barrio et al. (2018), Da Ré et al. (2022) or Pailos and Da Ré (2023).

[6]Indeed, in (Carnielli and Rodrigues 2019a), they already recognize that BLE and LET_J can receive an alethic interpretation. Moreover, in Rodrigues and Carnielli (2022), they argue that the junction of LET_J and the evidence interpretation is anti-dialetheist because "the simultaneous truth and falsity of *A* is expressed by $A \wedge \neg A \wedge \circ A$, and the latter yields triviality" (Rodrigues and Carnielli 2022, p.325). Here we will not evaluate this latter point. But Rodrigues and Carnielli's response suggests the soundness of the core of Barrio and Barrio and Da Ré's objections.

[7]In the paper (Arenhart 2020), Arenhart argues that Carnielli and Rodrigues' proposal fails in two ways: (i) evidence fails in doing justice to paraconsistent logics, because most of these systems validate the law of excluded middle, and (ii) the notion of evidence does not seem to adequately formalized by a paraconsistent logic. In what respects the second problem, Lo Guercio and Szmuc (2018) claim that the rule of the introduction of the conjunction of the systems BLE and LET_J is not compatible with the informal notion of evidence. We refer the reader to both papers for a more detailed presentation of these objections.

[8]We thank an anonymous reviewer for pointing this out to us.

[9]Notice that we are not arguing that given a pure system, for every application, there is more than one plausible philosophical interpretation. For instance, if we apply classical logic to electric circuits, maybe there is no more than one interesting conceptual account to elaborate. However, our point is more modest. Since we want to show that philosophical interpretations and applications are distinguishable, we only need to show that there are some applications, for example, semantic paradoxes, such that the same pure logic admits more than one interesting philosophical interpretation. We thank an anonymous reviewer for clarifying this point.

[10]Moreover, ST has been generalized by (Barrio et al. 2020). A hierarchy of non-transitive logics that are classical and can be applied to semantical paradoxes (Barrio et al. 2021). Similar consideration about alethic or bilateralist interpretations can be elaborated on the full hierarchy.

[11]These observations can also be found in Skyrms 1978; Burgess 1999.

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