Issues in the Philosophy of Logic: an Unorthodox Approach

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Abstract

In this paper six of the most important issues in the philosophy of logic are examined from a standpoint that rejects the First Commandment of empiricist analytic philosophy, namely, Ockham's razor. Such a standpoint opens the door to the clarification of such fundamental issues and to possible new solutions to each of them.

As an absolute principle, which is what it
purports to be, Ockam's razor is the expression
of a philosophical castration complex.
Oswaldo Chateaubriand
Logical Forms, II, p. 376

1. Introduction

In this paper I will discuss some issues in the philosophy of logic and mathematics, and will offer some unorthodox solutions, that is, solutions that deviate substantially from the main trend in analytic philosophy. Some of the issues treated here originate in Skolem, Quine or Benacerraf, and have captivated philosophers of logic and mathematics for the last few decades. Since any of the issues discussed here has enough flesh to be the sole one treated in a philosophical paper, my discussion of the different issues will be as brief as possible. Thus, very little will be said about other approaches, except when their omission would affect the intelligibility of the paper. The issues here discussed are the following: (i) first-order versus second-order logic, (ii) the so-called Skolem Paradox, (iii) Benacerraf's challenge to Fregean Platonism, (iv) Benacerraf's epistemological challenge, and (v) Quine's two-headed critique of Carnap's definition of analyticity, namely, the challenge to a definition of analyticity and the challenge to a definition of meaning and synonymy. All those problems, as well as their usual treatments and

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purported solutions, have a common source in the general empiricist dogma that permeates basically all Anglo-american analytic philosophy from Russell through Quine to Benacerraf, Field and Maddy. That dogma is frequently hidden in the appeal, as to a higher unquestioned authority, to Ockam's razor, according to which entities should not be proliferated unnecessarily. Certainly, one or another sort of empiricism and anti-Platonism has being present in the main trend of analytic philosophy since Russell. It must, however, be pointed out not only that the so-called father of analytic philosophy, Gottlob Frege, had nothing to do with the application of logic and mathematical tools at the service of empiricist ideologies, but, moreover, that logic and mathematics in no way favour empiricism or nominalism. Quite on the contrary, it can be shown that they require the existence of abstract entities and a non-empiricist epistemology.

2. Second-order Logic versus First-order Logic

The attempt to confine logical research to first-order logic is a glaring example of the entrenchment of the anti-Platonist dogma usually expressed by an appeal to Ockam's razor. Frege had no qualms in accepting a sort of higher-order logic, though the hierarchy in his logical system concerned only functions and relations, that is, non-saturated entities, not the saturated entities which he called 'objects', all of which collapsed in the first level of objects, what together with the requirement that functions be defined for all objects, was responsible for the Zermelo–Russell Paradox — as it should appropriately be called. Russell and Whitehead, and Ramsey developed two sorts of higher-order logic, namely, ramified and simple type theory, but particularly Russell, tried to neutralize the ontological commitment of such higher-order logics.

However, it was Skolem and later, more emphatically, Quine who championed the movement to ostracize second-order and, in general, higher-order logic. The philosophical crux of the matter is that the quantification over classes, sets or properties would in some sense require a commitment to the existence of abstract entities, an acknowledgement of the triumph of Platonism. Hence some metalogical arguments have been displayed in order to exclude second (and higher)-order logic. I will mention and comment only on two of those arguments.[1] Thus, it has been correctly pointed out that (full) second-order logic is semantically incomplete, that is, that there is no set of axioms for second-order logic that would adequately mirror second-order semantics. However, though semantic completeness is a very desirable property of a logical system, it is by no means the only important property, and the use of this argument while ignoring other important properties seems not to be philosophically innocent. For example, categoricity of theories was earlier considered a very desirable property of theories, and the fact of the matter is that thanks to the enormous strength of second-order logic, one can obtain categorical theories of arithmetic and analysis in second-order logic, whereas that is impossible in first-order logic. In fact, first-order arithmetic is not even ω-categorical. Of course, the non-categoricity of first-order logic has been a blessing in disguise, since without it the Löwenheim–Skolem Theorems would not be valid for first-order logic and, thus, classical model theory, with its mathematical richness and beauty, would have been severely hampered. Furthermore, decidability has been also considered a very desirable property of formal systems, and the fact of the matter is that whereas propositional logic and monadic first-order logic are decidable, full first-order logic is not decidable. Thus, one could with equal right have excluded polyadic first-order logic from the realm of logic on the basis of its undecidability, and we would have probably returned to a sophisticated version of traditional logic. Philosophers like Quine have, of course, accepted first-order logic in spite of such a limitation, and have rejected second (and higher)-order logic in view of its semantic incompleteness, in spite of its decisive virtues of having much more expressive capabilities and of the categoricity of the second-order formulations of important mathematical theories. The ground both for the acceptance and the rejection is more ideological than logical.

Another frequent argument against second-order logic is even more vulnerable, namely, that, in contrast to first-order logic, second-order logic has a plurality of different semantics. Thus, there is (i) the full second-order semantics, in which in all models, for any n, there are as many n-adic relations (or functions) as there can be on the basis of the cardinality of the universe of objects, (ii) the Henkin semantics, which allows truncated models, and (iii) the many-sorted semantics. Since the last two are essentially equivalent, I will concentrate the discussion on the first two. The argument against second-order logic stemming from the presumed plurality of semantics is simply based on confusion. There is only one sort of semantics for full second-order logic, namely full second-order semantics. As already pointed out, and emphasized by the adversaries of second-order logic, second-order logic is incomplete and categorical. Hence, neither the Löwenheim–Skolem theorems — which are incompatible with categoricity — nor the compactness theorem — which is an immediate consequence of semantic completeness — are valid for second-order logic. On the other hand, if we use the Henkin semantics for second-order logic, we are able to prove semantic completeness, compactness and the Löwenhein–Skolem theorems — and, thus, the categoricity of second-order arithmetic and analysis disappears —, briefly, we obtain results incompatible with second-order logic. In fact, what we do when we introduce Henkin semantics for second-order logic is to interpret second-order logic in first-order logic. This interpretation, though semantic, is similar to the syntactic interpretation of first-order logic in propositional logic used to establish the consistency of first-order logic relative to propositional logic,[2] though should not be rendered as a sort of consistency proof of second-order logic relative to first-order logic.[3] Moreover, in virtue of Lindström's most celebrated Characterization theorem, any extension of first-order logic for which are valid the usual Löwenheim–Skolem theorem and either the Upward Löwenheim–Skolem theorem or the Compactness theorem is equivalent to first-order logic. Hence, Henkian semantics (or its equivalent many-sorted semantics) is not an alternative semantics for second-order logic, but a reduction of second-order logic to first-order logic. Therefore, it is not true that there are many semantics for full second order logic, but only one: the full second order semantics.

3. The So-Called Skolem Paradox

As is very well known, one of the most important metalogical results about first-order logic is the already mentioned Löwenheim–Skolem theorem, according to which any theory formulated in a countable first-order language that has an infinite model, also has a countable model. As a consequence of this theorem, if you formulate the theory of real numbers or the theory of sets in a countable first-order language, the theory will contain, besides the usual non-countable models, a countable model. In the case of set theory, this is the essential gist of the so-called Skolem Paradox. Skolem argued,[4] on the basis of such a presumed paradox, on behalf of a relativization of the notions of set theory to the language in which it is expressed — and, if one is a set-theoreticist and believes in the reduction of the whole of mathematics to set theory, then of the whole of mathematics. In the particular case of set theory, Skolem tried to explain the anomaly by saying that though one cannot find inside the model a bijection between the universe of the model and the set of natural numbers, outside of the model such a bijection is available. It is remarkable that such a tortuous way of explaining the so-called Skolem Paradox has found so many adepts among distinguished logicians and philosophers of mathematics. A by far more simple and direct way out would be to acknowledge that first-order logic, which is already inadequate for expressing the Dedekind–Peano axioms — as they should be called — and real analysis, is also incapable as a language for the more powerful set theory.

To make the last point clearer — in case that it were necessary —, let us consider the following fictitious situation. I have been informed that in some oriental languages, like Vietnamese, the intonation is so important in the spoken language that two words with completely different meanings could look and sound alike, especially to the eyes and ears of a foreigner, but the intonation used in spoken language makes it clear to the native speaker which of the two or three similar words is being used. Consider now a fictitious somewhat similar language that I will call 'Contradictese', in which the words 'true' and 'false' look completely alike, but are distinguished by native speakers by means of their different intonations. Let us suppose that mathematical research papers or books are written in such a language, and let us suppose that a foreign mathematician named not Skolem but Slokem learns to read the language in order to have first hand acquaintance of mathematical research in Contradictese. Let us also suppose that Slokem begins reading a book, let us say, on Abelian groups. Since Contradictese does not distinguish in written language between true and false — or between assertion and negation —, Slokem very soon founds out that there are presumed contradictions in the formulation of Abelian group theory in Contradictese, contradictions that are not present when the same theory is expressed in Spanish or in Swedish. Since Slokem is a distant 'spiritual' relative of Skolem, he concludes that the consistency of Abelian group theory is relative to the language in which it is formulated and, moreover, than when formulated in Contradictese there is a contradiction from the inside, but seen from the perspective of Swedish or Spanish the contradiction is not present. If Slokem did not have some sort of prejudice against any other language besides Contradictese, he could very well have arrived to the much simpler conclusion that Contradictese is inadequate as a language in which to formulate Abelian group theory or, in fact, any other mathematical theory, instead of arguing for the relativization to a language of the consistency or inconsistency of a mathematical theory. In the same vein, if Skolem and his followers had no prejudice against second (and higher)-order logic, they would simply had concluded from the so-called Skolem Paradox that first-order languages are inadequate for set theory. In fact the so-called Skolem Paradox is nothing else than a consequence of the prejudice against second (and higher)-

order logic. For philosophers of logic and mathematics not afraid of higher-order logic the so-called Skolem Paradox has less reality than Homer's gods.

4. Benaccerraf's Challenge to Fregean Platonism

In his influential paper 'What numbers could not be' (1965), Paul Benacerraf has argued that since there are different, in principle indefinitely many incompatible ways of representing numbers in set theory, numbers cannot be sets. Moreover, he argues that numbers are not objects but simple point holders in a progression — or, in current parlance, ω-sequence —, and that arithmetic is not the study of the natural numbers but of all ω-sequences, one of which happens to be what we usually call the sequence of natural numbers. Thus, arithmetic is not ontologically committed to the existence of any object.

Before continuing, it should be pointed out that Benacerraf's ontological challenge described above is not a challenge to all sorts of Platonism, but only to those sorts, like Fregean Platonism or set-theoreticism, which are also reductionisms. In fact, Benacerraf's argumentation should be divided in three unequal parts. Firstly, the argument purporting to show that numbers are not sets; secondly, the argumentation purporting to show that arithmetic is the study of ω-sequences; and thirdly, the presumed corollary that purports to conclude that numbers do not exist. Let us follow such a division.

Benacerraf's argumentation has more weak than strong points, and in the best of cases only shows what non-reductionists have always known, namely, that numbers are not sets. The first part of his argumentation only shows that since one can "identify" numbers in set theory with sets in many incompatible ways, for example, the number '2' can be identified either with $\{\{\emptyset\}\}$ or with $\{\emptyset, \{\emptyset\}\}$, and the identifying sets have incompatible properties, for example, in the first one the number '2' is the unit set of the number '1' but not in the second, then numbers cannot be sets. However, a non-reductionist Platonist can perfectly well accept such a conclusion, without in any way seeing a menace to his contention that numbers are objects and arithmetic is ontologically committed. More exactly, for him, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are just different ways in which one tries in set theory to represent the number '2', in a very similar way to that in which two different, and sometimes very distant, senses serve to refer to an object. For example, 'the secretary of Trotsky for almost a decade' and 'the editor of *From Frege to Gödel*' are two different senses that refer to the man Jean van Heijenoort, and they are so far apart that one could very well use Jean van Heijenoort's case as an excellent example to argue on behalf of Kripke's contention in 'A Puzzle about Belief' (Kripke 1979). However, the man Jean van Heijenoort is not identified with any of those two properties attributed to him, but represented by them, in the same way in which 'the morning star' and 'the evening star' represent properties attributed to Venus, but are not identical with the planet Venus, which in fact is no star at all. Remembering Frege's appropriate metaphor in 'Über Sinn und Bedeutung' (1892, reprinted in Frege 1967) the two images of the moon in two different telescopes are two objective representations of the moon — in contrast to the subjective representations in our different retinas —, but they are not identical with the moon. A similar situation is present in Benacerraf's example: there are many set-theoretic representations of each natural number, but none of them is identical with the natural number being represented.

With respect to Benacerraf's second contention, it should be pointed out that the fact that the mathematician can study in a more general setting a class of isomorphic structures, or even of non-isomorphic but somehow formally related structures, does not mean that he cannot study the more concrete structure. Certainly, in virtue of the generalizations of the more familiar and concrete structures obtained during the nineteenth century and the first decades of the twentieth century, the mathematician can very well study n-dimensional Euclidean (or non-Euclidean) manifolds. However, that is in no way incompatible with the study of plane or solid Euclidean geometry in school. One should distinguish the study of geometry in the traditional sense from the study of such generalized geometries of n-dimensions, in which, for example, the parallel postulate could very well be replaced with a postulate incompatible with it. The same happens with arithmetic. There is a more concrete arithmetic of the natural numbers and at the same time it is perfectly feasible to study generalized arithmetics, in which the mathematician considers not only any structure isomorphic to the structure of the natural numbers but others like cardinal and ordinal arithmetic neither isomorphic to usual arithmetic nor isomorphic between them. In one case we are considering an intended, concrete and especially important structure, in the other case its possible generalizations. In a similar vein the mathematical physicist studies the structure of our universe, whereas the mathematician studies four-dimensional semi- Riemannian manifolds, of which our universe is presumably just an example. The mathematical physicist does not loose any interest in the study of the spatiotemporal structure of the universe just because there is such possible abstract treatment.

This last point brings us immediately to Benacerraf's last contention, namely, that numbers do not exist. If Benacerraf were consistent with his argumentation — which he is not —, he should derive the same conclusion in the case of our universe that he derives in the case of numbers. Since it is possible to study the abstract mathematical structure of which our spatiotemporal universe is just a special case, Benacerraf should conclude that our spatiotemporal world does not exist. But he does not derive such a conclusion from this perfectly structurally similar argument. The difference in his treatment of the perfectly similar situations lies not in any logical-mathematical distinction, but in his prejudice against abstract entities. Our universe has a material physical existence, with causal relations, in which Benacerraf believes, whereas the natural numbers are not material, physical objects causally related between them or with us. Thus, comes to the fore the underlying prejudice against abstract entities that is the hidden premise of Benacerraf's whole argumentation. He has not proven anything, but simply disguised as the conclusion of his argumentation what was from the beginning his most basic but hidden assumption. Therefore, he is in no way justified to conclude that numbers are not objects. Now, let us examine Benacerraf's other challenge.

5. Benacerraf's Epistemological Challenge

In his paper 'Mathematical truth' (1973) Benacerraf presents another sort of challenge against Platonism, this time an epistemological challenge. According to Benacerraf, a philosophy of mathematics needs to fulfil two very disparate requirements. Firstly, it must be compatible with Tarskian semantics. Secondly, it should be compatible with a causal epistemology, that is, the objects that are presumably known by us have in some sense to affect us causally. Those two requirements are disparate in more than one sense. First of all, Tarskian semantics is generally accepted as a central piece of our scientific knowledge about formal languages, namely, the semantics of formal languages, and is at the basis of a very well established and extremely fruitful area of logical research: model theory. Moreover, as clearly stated at the beginning of his seminal monograph 'The Concept of Truth in Formalized Languages'[5] and elsewhere, Tarski's intuitive motivation is the classical Aristotelian concept of truth, to which he wants to offer a rigorous and precise definition. In fact, that correspondentist motivation is clearly moulded in his Convention T, which is a necessary condition for any plausible truth definition, a material adequacy condition for any purported definition. Furthermore, since Tarski's definition of truth is successful — in fact, only — in the case of formalized languages, it is perfectly clear that, no matter how formalist were Tarski's philosophical leanings, his scientific work on the concept of truth is concerned with abstract entities. In the invaluably rich offspring of that definition, namely, model theory, the concern with abstract, causally inert, entities, which are also completely independent of our constructions, is even more evident.[6] Hence, Benacerraf's two requirements on philosophies of mathematics are not only very disparate, but they are hardly compatible.

On the other hand, if we take seriously Benacerraf's second requirement of a causal connection between the entities known and the knower, then, as has already been pointed out by many,[7] a significant part of current physics would not satisfy such a requirement. Thus, one would have to deny existence to most entities postulated by microphysics, since they are not causally related to us in any way, but simply introduced in order to give coherence to some theories and explain some physical phenomena. Moreover, also in cosmology coherent theories are developed, which imply the existence of entities with which we are not causally related. Therefore, if physics were to obey Benacerraf's second requirement — which is, incidentally, a derived version of Ockam's razor — substantial parts of current physics would have to be rejected as being pure metaphysical speculation. Of course, Benacerraf and his followers have not intended to exclude such parts of physics from serious scientific research. But as happened with the criterion of meaning of complete verification designed by logical empiricists in order to exclude metaphysical statements as devoid of meaning, the baby was also thrown away with the placenta.

Furthermore, the fact that Benacerraf's second requirement was not intended to exclude physical entities, but only mathematical ones brings to the fore once more the anti-Platonist prejudices hidden behind such a requirement. It is clear that abstract entities — *per definitionem* — do not causally affect the epistemological subject, they do not act on our senses or our nervous system, or blind us as the sun does when we look at it. However, we can ask ourselves whether what we see in current day perception are pure sensations. The answer to that question is clearly no. When we see a book or a tree, we do not have first sensations of the green colour of the leaves, or of the red colour and the hardness of the cover of the book and then compose those sensations to form an object. It is precisely the opposite: we see the tree or the book, and then we could abstract the green colour of the leaves or the red cover of the book. As Husserl already knew —

and the young Carnap of *Aufbau* repeated — we are not acquainted directly with sensations, but we abstract them from what is primarily given to us. Moreover, as Husserl has also stressed, what are primarily given to us are not even isolated objects, the red book or the tree, but states of affairs. I see the red book on my desk at the side of some notebooks. I see my personal computer on which I am presently working on the smaller desk at the side of the printer. If I did not see it on the desk, but as an isolated object, I would be certainly worried that it could fall to the floor. What is given in our perceptions are, hence, not even objects, but states of affairs. Furthermore, states of affairs contain non-sensible elements, that is, constituents that correspond to the words 'at the side of' or 'on', which are not sensibly given. Nonetheless, they are present in perception and even affect our judgements about perception. Consider, for example, the statements 'Peter and John are at the door' and 'Peter or John is at the door'. If when I come to the door I see only John, I (and you also) would consider the first statement false and the second true, though the difference between the two statements — besides a minor grammatical adjustment — lies in the presence of the particle 'and' in one case and 'or' in the other, to none of which there corresponds any sensible correlate and, *a fortiori* no sensible correlate causally related to us. In the same vein, if someone asserts that the book is on the desk, but it is in fact under the desk, we would consider such an assertion false. In both examples, non-sensible components of perceptions play such a decisive role in sense perception as to be determinant for the truth or falsity of statements about what is sensibly perceived. Therefore, there are abstract components in simple sense perception, and without them our knowledge of the world would be extremely primitive. Moreover, even to understand the causal connections between events, knowledge of states of affairs with their abstract components is presupposed. Thus, if I did not perceive the computer on the desk or on the table while writing, I would hardly understand that it has not fallen to the floor. Furthermore, sets or collections of objects are also abstract constituents present in our current perceptions. There is no sensible correlate to the word 'set' when I state that I perceive a set of pens on my desk, and, certainly, I distinguish it perfectly well from the extensive sum or whole of the pens put together. If I were to apply enough heat to the pens in order that they all melt together, I would not consider that I were perceiving a set of pens, but only an extensive whole from material that once belonged to the different pens. Hence, these rudiments of Husserlian epistemology[8] do not have anything to do with Penelope Maddy's confused presumed epistemology of abstract entities, which by presupposing the correctness of Benacerraf's second requirement, could not take off the floor of sensible objects, not being able even to distinguish between a set and the extensive whole of its members, nor between a unit set and its only member.[9]

6. On Quine, Meaning and Synonymy

In his extremely popular paper 'Two Dogmas of Empiricism' (Quine 1951), Willard van Orman Quine submitted Carnap's notion of analyticity to a devastating criticism. He showed that Carnap's notion of analyticity was so entangled with the notions of meaning and synonymy that every attempt at defining one of the three notions would be circular. Quine concluded that philosophers have to abandon the notion of analyticity and *a fortiori* the distinction between analytic and synthetic statements. Moreover, philosophers should abandon the pursuit of a definition of the meaning of isolated statements. Thus, he propounded to search for the meaning of statements only in the context of our whole theory or, better, our whole web of beliefs. In this web of beliefs there is no essential or qualitative difference between statements, but just a matter of degree. Logical and mathematical statements lie in the centre of our web of beliefs, and since their abandonment would require radical changes in our web of beliefs, we tend to protect them from any refutation and try to make adjustments in the more peripheral statements of that web. Nonetheless, the difference is not qualitative but a matter of degree, and it could very well happen that in case of serious clashes between theory and experience we would reasonably opt to abandon a logical or mathematical law. Such a holistic conception has been baptized in the specialized literature as the Duhem–Quine thesis, though Duhem never sustained such a radical holistic thesis. On the contrary, Duhem just argued that in the most basic fundamental empirical science, that is, physics, it is impossible to submit an isolated hypothesis to experience and so to refute it in case experiments contradict the empirical consequences derived from the hypothesis. The grounding for Duhem's contention is that in order to make the experiment the physicist has to presuppose the validity of other unquestioned underlying physical laws and, hence, it is not excluded that one or more of such underlying laws are responsible for the negative empirical result, not the hypothesis under discussion. Therefore, Duhem concludes that it is impossible to make crucial experiments in physics, since the observable consequence submitted to experience has been obtained not from the hypothesis alone, but from a whole group of physical laws and hypotheses, and there is no logical argument that would allow us to conclude that precisely the hypothesis in question is the sole responsible of the refuted empirical consequence.

In this and the following sections I will be concerned with two different aspects of Quine's conclusion. In the next section I will be concerned exclusively with the notion of analyticity, though by no means with an attempt to vindicate Carnap's particular notion of analyticity, whereas in the present one I will discuss Quine's contention that the notions of meaning and synonymy are undefinable and that both are irremediably entangled in a circle with Carnap's notion of analyticity. Thus, I will sketch how one could try to define meaning and synonymy without any reference to analyticity.

Let us first limit the discussion exclusively to logical-mathematical statements. Let us consider transformations of logical-mathematical statements into logical-mathematical statements such that true mathematical statements are always transformed into true ones and false logical-mathematical statements are transformed into false ones. It is easy to see that the set of transformations that assign true statements to true statements constitutes a group of transformations. It clearly contains the identity transformation; for any transformation f it contains its inverse; and for any pair of transformations it contains their concatenation, and the operation of concatenation is associative. Moreover, as can be easily seen, the set of transformations from false logical-mathematical statements to false logical-mathematical statements also constitutes a group of transformations. Although I presupposed the notions of 'true' and 'false' in the above discussion, it should be stressed that they could be seen as the two equivalence or abstraction classes obtained from the two distinct groups of transformations considered.

Let us now consider more restricted sets of transformations between logical-mathematical statements, namely, such by means of which interderivable statements are transformed into each other. Thus, for example, The Axiom of Choice is transformed into Tychonoff's Theorem, but neither into '$2 + 3 = 5$' nor into the Principle of Non-Contradiction. As can be easily seen, those sets of transformations also constitute groups of transformations of logical-mathematical statements, since they contain the identity transformation, for any transformation f they contain its inverse, and they can be concatenated and the operation of concatenation is associative. The equivalence classes abstracted from such groups of transformations can be called, using Husserlian terminology, situations of affairs.

We can, however, consider more restricted sets of transformations between logical-mathematical statements, namely, such that transform statements like the Ultrafilter Theorem, that is, 'Every filter can be extended to an ultralilter' into 'Every filter can be extended to a maximal dual ideal', or a statement like '5 + 2 = 7' into '8 − 1 = 7', thus, any statement into one that differs from it by the replacement of one or more constituent parts by names of the same entities. It can be easily seen that once more the different sets of transformations constitute groups of transformations. However, contrary to what occurred in the other two families of groups of transformations, The Axiom of Choice and Tychonoff's Theorem do not belong to the same group of transformations. Following Husserlian terminology, we can call the equivalence classes abstracted from such groups of transformations 'states of affairs'. Since the groups of transformations that preserve states of affairs are proper normal subgroups of groups of transformations that preserve situations of affairs, and the latter are proper normal subgroups of the groups of transformations that only preserve truth value, one can see situations of affairs as equivalence classes of states of affairs and truth values as equivalence classes of situations of affairs.

Furthermore, we can now consider solely the identity transformation between logical-mathematical statements. It also constitutes a group of transformations, namely, the trivial group, which is a proper subgroup of any non-trivial group of transformations. Finally, in case there exists a proper non-trivial subgroup of a group of transformations that preserves state of affairs, we can define the meaning of a statement as its equivalence class under such a group of transformations. In case there is no such non-trivial subgroup, we define meaning as the equivalence classes of the trivial subgroup. In this last case, each equivalence class is a unit class and there are no synonyms in the language. In the former case, some of the equivalence classes contain more than one member and we say that two statements are synonymous exactly when they belong to the same meaning.

From the definitions of meaning and synonymy for statements, one could then obtain the definitions of synonymy of their constituent parts and of meaning of their constituent parts. One can apply the same procedure as above to obtain corresponding definitions of meaning and synonymy for statements and their constituent parts in natural language. Of course, at first sight, the corresponding notions of groups of transformations seem not as sharp as for logical-mathematical statements, especially those from which the situations of affairs are abstracted, since in the case of natural language we cannot base the group of transformations on interderivability. Nonetheless, such groups of transformations also exist for natural languages. Thus, the statements 'Joe bought a car on 12 July 2007' and 'Jeff sold a car on 12 July 2007' could very well belong to the same group of transformations modulo situation of affairs in case it is a fact that "Jeff sold a car to Joe on 12 July 2007'. But in any case, one could sidestep such groups of transformations in the case of natural language without affecting the result of defining meaning and synonymy.

7. On Analyticity

Most analytic philosophers believe that Quine's critique of Carnap's definition of analyticity has eradicated that notion and, moreover, the distinction between analytic and synthetic statements once and for all from the philosophical scenario. Such a belief, however, is just an indication of the narrow philosophical views of the believer. First of all, one does not need Quine's criticism of Carnap's notion of analyticity to know that it was misguided. To consider analytic such statements as 'All bachelors are unmarried', which are clearly either the result of a convention or, more probably, the result of the historical and, thus, empirical evolution of language, is a philosophical aberration. But in any case, Carnap's definition of analyticity, which is an offspring of Kant's characterization of analytic judgements (read: statements) as those in which the concept of the predicate is already contained in the concept of the subject, is not the only notion of analyticity propounded by important philosophers in recent times. It is not even the only offspring of Kant's views. Indeed, Kant had two non-equivalent characterizations of analyticity, the second being that a judgement (statement) is analytic if it can be derived from the Principle of Non-Contradiction (which probably was meant to include also the Principle of Identity and the Principle of the Excluded Middle) (see Kant's *Kritik der reinen Verfnunft*). An offspring of this second Kantian characterization of analyticity is Frege's definition of analyticity in *Die Grundlagen der Arithmetik*, according to which a statement is analytic if it can be derived from logical axioms by means of logical rules and definitions. Such a definition is certainly not dependent in any way on the notions of meaning and synonymy, and is, thus, not vulnerable to Quine's criticism of analyticity. Moreover, it should be perfectly clear that statements like 'All bachelors are unmarried men' are not analytic in virtue of Frege's definition. Nonetheless, Frege's definition is vulnerable to other sort of criticism. Due to the failure of Frege's, Whitehead's and Russell's and any other attempt to derive mathematics from logic, briefly, due to the failure of logicism, on the basis of Frege's definition of analyticity, logical statements would practically be analytic *per* definitionem, whereas a substantial part of mathematical statements would have to be classified as synthetic, be it as synthetic *a priori*, as Kant thought, or as synthetic *a posteriori*, that is, empirical, as Mill, Quine and many others in the Angloamerican philosophical tradition have believed. However, if something has been learnt from more than a century of studies in the foundations of mathematics is that logic and mathematics are so narrow relatives — if not mother and daughter, as argued by logicists, then sister disciplines —, that if one is analytic, the other also is, and if one is synthetic *a priori* or synthetic *a posteriori*, the other also is. Therefore, either one accepts Quine's contention that all statements are empirical, or one accepts a sort of radicalized Kantianism, which would also consider logic as synthetic *a priori*, or one has to look for a more adequate definition of analyticity.

Fortunately, there is in the most fundamental philosophical literature — though one not usually read by analytic philosophers — a much more promising definition of analyticity, namely, that of Husserl in his *opus magnum*, *Logische Untersuchungen*.[10] According to Husserl, a general statement or law is analytic if it is true and remains true if all its material constituents are eliminated, that is, if the general statement remains true after being completely formalized. Instantiations of such general laws are called by Husserl analytically necessary propositions (or statements), being true also in virtue of their form, not in virtue of any material content. Thus, Husserl's definition of analyticity — which is an offspring not of Kant's views but of Bolzano's — is based on the logical form of statements. Certainly, logical and mathematical laws seem to be true in virtue of their form, and should be considered analytic in Husserl's sense. Moreover, statements like 'All bachelors are unmarried men' are clearly not analytic in Husserl's sense, since they are not formalizable *salva veritate*. Nonetheless, there seem to be two very disparate sorts of statements that intuitively should be analytic, but fail to be so in virtue of Husserl's definitions. The first sort is that of metalogical statements, like the Compactness theorem, the (Semantic) Completeness theorem or the Upward and Downward Löwenheim–Skolem theorems. The second sort of statements are relatively 'concrete' statements in number theory. Thus, for example, the equation '$1^3 + 2^3 + 3^3 + 4^3 = 100$' is not formalizable *salva veritate*.

Husserl's definition of analyticity is not only based on logical form, but is a syntactical one. I will now offer a different definition of analyticity, which is also concerned with form, but is a semantic one. It is based on the most general notion of isomorphism, which can be defined for models of any logical language. Informally, we can say that two structures $\mathfrak{A}$ and $\mathfrak{B}$ for a logic L are isomorphic if (i) there is a bijective correspondence h between the members of the universes of the two structures, (ii) for any operation f in $\mathfrak{A}$, the corresponding operation f^* in $\mathfrak{B}$ and any $a_1, \ldots, a_n$, a belonging to the universe of $\mathfrak{A}$, $h(f(a_1, \ldots, a_m) = a) = f^*(h(a_1), \ldots, h(a_n)) = h(a)$, and (iii) for any relation R in $\mathfrak{A}$ there is a relation R^* in $\mathfrak{B}$ such that when $a_1, \ldots, a_n$ belong to the universe of the first structure, they are in the relation R exactly when their images under h are in the corresponding relation R^*. One can now define analyticity in the following way: a statement S is analytic if (i) S is true in at least one structure and (ii) if S is true in a structure $\mathfrak{M}$, then it is true in at least any structure isomorphic to $\mathfrak{M}$.

It should be stressed that in virtue of the above definition, it is not excluded that an analytic statement be true in structures not isomorphic to structures in which it is true. Hence, if we formulate the Dedekind–Peano axioms in first order logic, in which case they are not categorical and, thus, admit both semi-standard models of uncountable cardinality (that is, uncountable extensions of the standard model) and non-standard countable and uncountable models, it does not affect the analyticity of such axioms. Moreover, though Skolem's statement is not true in the standard model, since it is true in at least any structure isomorphic to Skolem's non-standard model, it is also an analytic statement in virtue of the above definition. On the other hand, it is perfectly possible that an analytic statement be false in a structure not isomorphic to a structure in which it is true. Thus, analytic statements need not be true in all possible structures and, hence, analyticity does not coincide with the intuitive notion of logical truth — no matter how we make precise such an elusive notion.[11] All logically true statements are analytic, but there exist analytic statements that are not logically true. Moreover, since we can consider logical languages as linguistic structures, and the metatheoretic results considered above, when valid for a language $\mathcal{L}$, are also valid for any language $\mathcal{L}^*$ isomorphic to $\mathcal{L}$, such metatheoretic statements — like the Compactness theorem, the Completeness theorem, etc. — are all analytic in virtue of the above definition, since they are true in any language isomorphic to a first-order language in which they are true. Finally, number-theoretic statements, like '$1^3 + 2^3 + 3^3 + 4^3 = 100$', are also analytic under the above definition, since they are true in any structure isomorphic to that of the natural numbers. Therefore, Husserl's syntactic notion of analyticity is more restrictive than the semantic one I have offered, since its extension is a proper subset of the extension of the latter.[12]

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Keywords
Ockham's razor, second order logic, Skolem's Paradox, Benacerraf, Quine, analyticity.

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Resumo

Neste artigo, seis das mais importantes questões na filosofia da lógica são examinadas de um ponto de vista que rejeita o Primeiro Mandamento da filosofia analítica empirista, a saber, a navalha de Ockham. Um tal ponto de vista abre a porta ao esclarecimento de tais questões fundamentais e possíveis novas soluções a cada uma delas.

Palavras-chave
Navalha de Ockham, lógica de segunda ordem, paradoxo de Skolem, Benacerraf, Quine, analiticidade.

Notes

[1] For a detailed discussion of arguments for and against second (and higher) order logic, see the papers by Boolos, Jané, Resnik, Shapiro, Simmons, Tharp and Wagner in Shapiro 1996, as well as Otávio Bueno's excellent paper 'A Defence of Second-order Logic' and both volumes of Oswaldo Chateaubriand's outstanding book *Logical Forms*. Incidentally, though, as pointed out, I would not dare to express the views of other Latinamerican philosophers on any of the issues discussed in the present paper, on this particular issue

of second versus first order logic my views coincide with those of the two distinguished Brazilian philosophers.

[2] For the use of a syntactical interpretation in propositional logic of first-order logic in the relative consistency proof of the latter to the former, see, for example, Elliot Mendelson's classic *Intoduction to Mathematical Logic*, second edition, p. 62, or Robert Rogers' excellent informal book *Mathematical Logic and Formalized Theories*, pp. 47–8.

[3] Of course, there are more reliable ways to establish the consistency of second-order logic, for example, by obtaining it as a corollary of Tarski's definition of truth applied to second-order logic, or simply by observing that if second-order logic were inconsistent, all statements expressible in second-order logic would be derivable from its set of inconsistent axioms and, hence, all true second-order statements would be derivable. However, since there is no semantically complete set of axioms for second-order logic, it follows that second order logic is consistent.

[4] See, for example, his paper 'Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre' 1922, translated into English in van Heijenoort 1967, pp. 290–301. See also, for example, the papers by Paul Benacerraf and Crispin Wright in Shapiro's already mentioned book *The Limits of Logic* (Shapiro 1996).

[5] The paper is a translation of Tarski's 'Der Wahrheitsbegriff in den formalisierten Sprachen' 1935, and is the central piece of Tarski 1956. See also his informal expository paper 'The Semantic Conception of Truth and the Foundations of Semantics'.

[6] See on this issue Wilfrid Hodges criticism of Putnam's 'Models and Reality' on p. 34 of his monograph 'Elementary Logic' (Hodges 1983).

[7] On this issue, see, for example, James Robert Brown's paper 'π in The Sky' (Brown 1990), as well as the present author's 'On Antiplatonism and its Dogmas", 1996, reprinted in Hill and Rosado Haddock 2000, pp. 263–89.

[8] For Husserl's epistemology of mathematics, see *Logische Untersuchungen* (Husserl 1900–01). See also the present author's paper 'Husserl's Epistemology of Mathematics and the Foundation of Platonism in Mathematics' 1987, reprinted in Hill and Rosado Haddock 2000, pp. 221–39.

[9] For Maddy's views on our perception of abstract entities and her distortion of Gödel's views, see, for example, Chapter Two of her book *Realism in Mathematics*, 1990.

[10] For Husserl's definitions of analytic law and analytic necessity, see *Logische Untersuchungen*, U III, §12.

[11] Interestingly, recently the distinguished Brazilian philosopher Oswaldo Chateaubriand has offered in the second volume of his already mentioned book *Logical Forms*, without being aware of Husserl's definition of analyticity, a characterization of logical truth and its instantiations in virtue of their logical form, which is very similar to Husserl definition of analytic law and and analytically necessary propositions. I coincide with Chateaubriand in considedring such a sort of definition in virtue of the logical form of statements more

appropriate as a definition of logical truth. Nonetheless, the unclarity of our present views on what should be considered logic amounts to a serious difficulty of such a definition.

[12] For a more detailed discussion of Husserl's and the present author's definitions of analyticity, see my forthcoming paper 'Husserl on Analyticity and Beyond' as well as Chapter 4 of my forthcoming book *The Young Carnap's Unknown Master*.