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On the Structure of Rationality in the Thought and Invention or Creation of Physical Theories (with a special attention to the cases of general relativity and of quantum theory)

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Abstract. We want to consider anew the question, which is recurrent along the history of philosophy, of the relationship between rationality and mathematics, by inquiring to which extent the structuration of rationality, which ensures the unity of its function under a variety of forms (and even according to an evolution of these forms), could be considered as homeomorphic with that of mathematical thought, taken in its movement and made concrete in its theories. This idea, which is as old as philosophy itself, although it has not been dominant, has still been present to some degree in the thought of modern science, in Descartes as well as in Kant, Poincaré or Einstein (and a few other scientists and philosophers). It has been often harshly questioned, notably in the contemporaneous period, due to the failure of the logistic programme, as well as to the variety of "empirical" knowledges, and, in a general way, to the character of knowledges that show them as transitory, evolutive and mind-built. However, the analysis of scientific thought through its inventive and creative processes leads to characterize this thought as a type of rational form whose configurations can be detailed rather precisely. In this work we shall propose, first, a quick sketch of some philosophical requirements for such a research programme, among which the need for an harmonization, and even a conciliation, between the notions of rational (or rationality), of intuitive grasp and of creative thought. Then we shall examine some processes of creative scientific thought bearing on the knowledge and the understanding of the world, distinct from mathematics although keeping tight relations with them. Contemporary physical theories are privileged witnesses in this respect, for in them the rational thought of phenomena makes an intrinsic use of mathematical thought, which contributes to the structuration of the formers and to the expression of their concepts (which entails the physical contents of the latter). The General Theory of Relativity and the Quantum Theory are exemplar to this, as they directly reveal what can be called the "drag of physical thought par the mathematical form", which makes possible to overcome the limitations of the physical knowledge previously adquired. This process is tightly related to the modalities and to the stucture of the rational thought underlying it. This is what we would like to show.

Keywords: Creative thought; general theory of relativity; intuitive grasp; mathematical form; logistic foundational program; mathematical thought; physical content; physical thought; quantum theory; scientific knowledge; scientific thought; structuration of the rational.

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This work is dedicated to Newton da Costa for his eightieth birthday (and some months that have lapsed since the Campinas Symposium which to my deep regret I could not attend).

My dear Newton, I see you still as young and intellectually productive as some twenty five years ago, when we came to know each other. I hope you will like the present paper, prepared for you, and which tries to illustrate and argument that the physical thought as well as the mathematical thought are creators of forms with meaningful contents and always in becoming; that in them, reason goes together with intellectual intuition and invention, all this movement having its source in the mind's desire and formulated exigency of intelligibility. By your work and attitude you are a vivid example for many of this inextinguishable quest.

1. Introduction and outlook

I would like to consider anew the question, which is recurrent along the history of philosophy, of the relationship between rationality and mathematics, by inquiring to which extent the structuration of rationality, which ensures the unity of its function under a variety of forms (and even according to an evolution of these forms), could be considered as homeomorphic with that of mathematical thought, taken in its movement and made concrete in its theories. This idea, which is as old as philosophy itself, although it has not been dominant, has still been present to some degree in the thought of modern science of Descartes as well as of Kant, Poincaré and Einstein (and a few other scientists and philosophers). It has been often harshly questioned, notably in the contemporaneous period, due to the failure of the logistic programme, as well as to the variety of "empirical" knowledges, and, in a general way, to the character of knowledges that show them as transitory, evolutive and mind-built. However, the analysis of scientific thought through its inventive and creative processes leads to characterize this thought as a type of rational form whose configurations can be detailed rather precisely.

I shall first quickly recall some philosophical requirements for such a research programme, among which the need for an harmonization, and even a conciliation, between the notions of *rational* (or *rationality*), of *intuitive grasp* and of *creative thought*. Then I shall examine some processes of creative scientific thought bearing on the knowledge and the understanding of the world, distinct from mathematics although keeping tight relations with them. Contemporary physical theories are privileged witnesses in this respect, for in them the rational thought of phenomena makes an intrinsic use of mathematical thought, which contributes to the structuration of the formers and to the expression of their concepts (which has effects on the physical contents of the latter).

The General Theory of Relativity and the Quantum Theory are exemplar of this, as they directly reveal what can be called the "dragging along of physical thought by the mathematical form", which made possible to overcome the limitations of the physical knowledge previously adquired by modifying the physical concepts and by creating new theoretical views in Physics. This process is tightly related to the modalities and to the structure of the rational thought underlying it. I hope to show somewhat by this that *scientific creative thinking*, which is at the core of the dynamics of scientific knowledge, can be approached in rational terms and stands therefore as one of the most important problems of Philosophy of Science, contrarily to a constraining claim, that has dominated throughout nearly a century, to reject it outside of Philosophy.

2. The relationship of Physics and Mathematics as seen from the dynamics of the working thought

One has never finished with the problem of the relationship between Physics and Mathematics. To enter into its deep reasons, we have to consider both sciences in their permanent renewals. Mathematics, as the science of "quantities", of abstract rational objects and forms, of "formal contents",[1] is growing continuousy. Physics, as the science of the inanimate objects and events in Nature, will most probably never be exhausted. And so will their relationship, that has been evoluting through History, before and after the constitution of Modern Physics, with the claim that physical concepts must have the form of quantities, in the same sense as Mathematics gives to this term, notwithstanding the fact that physical concepts are different in nature from mathematical ones, for they do not point at the same kind of existence or reality.

I would like to begin here with a particular point of view on this relationship which is a dynamical one: I mean dynamical as to the intellectual processes in the mind that deal with them (Mathematics and Physics) in scientific investigation. That this process is oriented towards intelligibility will be the *leimotiv* of this contribution. The relationship between Physics and Mathematics has often been conceived in an oversimplified, and actually distorted, way, as a relationship between "Mathematics and experience". In such a formulation, the proper role of Physics does not immediately appear, whence Physics is the science that addresses the knowledge of the domain of nature we aim at, and to which is referred what is designated as "experience". Experience, in the expression above, takes all the space, as if Physics — as knowledge of Nature — was reduced only to it. It would be difficult to envisage in such conditions a direct connection between Mathematics, which as a science is not related to Nature — if not in the origin of its constitution —, and experience by which we question this Nature through experiments. Such a questioning obviously presupposes ideas able to guide it, and to put in order and translate the answers — that is to say the results of the experiments, the data collected from experience. These ideas are nothing but concepts and theory (or theories) that organize(s) the concepts. Physics includes theory and experiment: it is in the formulation of physical theory that Mathematics come into play. But the physical theory is not for that reducible to its mathematical formulation: its concepts — expressed under the form of mathematical quantities — have as their meaning the physical content to which they refer, pointing fundamentally at phenomena of nature.[2]

One aspect of the question of the relationship between Physics and Mathematics concerns the form and structure of the various physical theories that can been considered — classical, relativistic, quantum, ... — with respect to the mathematical theories which are used respectively to express them — calculus of partial differential equations, tensor calculus, linear operators and Hilbert spaces, theory of transformation groups, etc. Both series have one to the other similarities and differences: the similarities lie in the formal expression of the quantities involved and their interrelationships within the theoretical structure; as to the differences, they hold essentially to the fact that physical theory refers to the phenomena of nature, while mathematics refer only to themselves.

One can discern two characteristic marks of the difference between a physical theory involving concepts expressed by quantities and the mathematical theory which expresses the formal properties possible a priori of these quantities. The first is that the *physical meaning* of the quantities implies constraints on their expression, which are imposed to their a priori mathematical form — for example, by submitting them to physical principles, which interprete general properties of nature. The second is the role of observation and experience in physics, which establishes in practice the connection of the theorico-conceptual representation with phenomena. On the one hand, experimental results provide *approximations* to the values attached to the considered quantities. Moreover, these results indicate or possibly impose constraints — be them already anticipated by the theory or new ones — for the quantities, which correspond to the "response" of nature to questions posed by the theoretical representation.

But this is not the only issue of this relationship which concerns us here. If it were only that, the title chosen for this paper would have been different, for example: "The nature of the relationship between Physics and Mathematics", and I would examine in detail for the various physical and mathematical theories, considered in their form, their structure and meaning, the implications of what has just been briefly stated. But what I want to do here, is to extend the inquiry of this relationship, generally considered for well established theories with stable formulation, to the processes of thought that elaborated them, in so far as it is possible to follow them significantly. With the *dynamical* character of the relationship of Physics with

Mathematics, on which we may hope by this way to obtain some lights, I have in mind the nature of the functioning and operation of scientific rationality, as thought in motion, producing knowledge. This goal may look ambitious, if considered in its generality, but we shall content ourselves here with a circumscribed and limited approach, since we will restrict ourselves to two already well beaconed directions of the development of contemporary Physics.

Considering the present state of the formulation of physical theories, we notice, after Einstein and others, that the relationship between the theoretical formulation of Physics (which incorporates the appropriate Mathematics) and the corresponding phenomena to which experience gives access is very indirect, marked by a greater and greater remoteness, when compared to the earlier physical theories such as the classical ones, between the theoretical-mathematical form and the physical content in phenomenic terms which this form serves to express and to get at. This is notably the case, as it is known, of Quantum Theory. This distance does not prevent the fertility of the theoretical representation: on the contrary, it appears to amplify it. We therefore expect from the dynamics of the theoretical thought of Physics that it bring some light on this apparent paradox, that can be expressed as: intelligibility of the physical (concrete) world is obtained by the development of an abstract symbolic thinking which is out of proportion (at least apparently) with what it aims at.

3. About invention as creation and rationality in the scientific thought

3.1. Has invention in science to do with rationality?

In what follows, therefore, attention will be focused on *invention* and *creation* of physical theories as constituting a process of bringing to light new knowledge. As it is known, the two selected theories, the Theory of Relativity (Special, but above all, General) and Quantum Theory (which culminates in the Quantum Theory of Fields), dominate the field of contemporary Physics: these are the two most powerful theories, by their descriptive and predictive ability, of the history of Physics. I am alluding here to the way they have permitted to learn, to conceive and to understand the phenomena of the physical world, which without them would have remained outside our knowledge. The mere statement of some of these phenomena is speaking in this regard. Regarding the General Theory of Relativity: the curvature of light rays in the vicinity of large masses, the slowing of clocks in gravitational fields, gravitational waves, black holes, and relativistic Cosmology that reveals the evolution of the Universe. Regarding the Quantum Theory: the diffraction of particles of matter, the exclusion principle, Bose-Einstein condensation, non-local separability and entanglement of interacting quantum systems, the superposition of quantum systems states, and decoherence, the Quantum Field Theories and their many consequences in the physics of particles and fields.

But this power is also effective in the unprecedented accuracy that these theories can achieve in the calculation of properties of physical systems or bodies. This fertility and this extreme precision characterise the close adequation of these theories to the phenomena, and this is usually the perspective which Philosophy of Science considers. One speaks of verification or falsification, of "application of Mathematics to Physics", of "mathematical formalism applied to empirical data", of the remarkable (and even "miraculous" or "enigmatic") "adequacy of mathematics to physical reality", up to invoking eventually Platonic Ideas. Undeniably, this aspect needs clarification, and I will say a few words of it, but it will not be the main purpose of this paper.

My main interest here will be to *invention*, in the thought of the authors of these theories, in their minds at work, searching to know better and to understand physical phenomena of Nature, by "building", so to speak, an intellectual picture or representation of them. To achieve this goal, they make use of adequately formed physico-mathematical notions — actually concepts, or entities and algorithms functioning as such — and theories (as the system of these notions or concepts), that look at first very distant from a representation of natural phenomena. (One may wonder, however, which kind of a priori view one could have of such physical phenomena and systems, if any, which are so distant of our perception?[3] Could their representation be of another kind than intellectual?). It is remarkable that, in these cases at least — without pronouncing here about generality —, *invention* (indeed, *creation*) has to do directly with *rationality,* on which *intelligibility* stands. As it is known, the dominant trends of philosophy in xxth century (namely logical positivism and empiricism, analytical philosophy, but also critical rationalism) were to reject as a matter of principle the question of scientific invention or creation, considered irrational and depending essentially on psychological processes — or, more recently, to decisions socially determined.

But if this was really the case, one would inquire by which strange random conjunction could an "irrationnal" (and "subjective" in this sense) process get *in the end* at such a highly *rational* result as the production of new theoretical elements and features which satisfy the requirement of rationality — for the obtained theory is rational, whatever the details and its eventual imperfections — and provide understanding. Such result answers, even if it is in unexpected terms, the question *initially* asked to himself by the inventor scientist as a *rational* one. For it seems clear for those interested in the actual practice of Science, and particularly of Mathematics and of mathematized Physics, that the work of the researcher's thought *starts* with posing a *problem*, formulated in *rational* terms and motivated by a rational concern

— the desire or exigency to *understand*, that is of *intelligibility*. Whatever be the intellectual journey followed from there by the searcher — a journey made of a series of reasonings and intuitions through mental states, some of them conscious and others semi-conscious, as Poincaré himself testimonied when evoking his own discovery of fuchsian functions —, the line that has been travelled through from the starting problem up to the finding of the solution, can in its gross features be qualified as a rational one, as it can be followed and traced according at least to some intelligibility features, which is what we are interested in here.

Of course, once the invention or dicovery has been made, it is possible to recast the form of the first obtained theory or result in order to make the whole more "economical" intellectually, more "logical" or more axiomatical. But such optimal "rational reconstruction", to use the words of Imre Lakatos (1978), is not a requirement for a new discovered knowledge be rationally legitimated, all the more because such knowledges have most often been set forth as the result of a "building-up" intellectual process, of a "construction" rationally motivated, and performed on rational grounds.

By insisting, as I did in what precedes, on the rationality inherent to the process of scientific invention, I lit thereby the specific wording of the title of this work, whose intended topics is very precisely "the structure of rationality in the invention ..." of physical theories such as the Theory of General Relativity and the Quantum Theory. Both theories have been regarded at the time of their presentation — and have been still considered up to now — as very different one from the other in all respects: as to their purpose, their structure, their modalities, their concepts and the type of quantity which they involve; in a word, as to the kind of description they provide of the phenomena of Nature. They bear respectively on two phenomenal areas of the natural world that are presently disconnected one from the other in our representations, and whose connection and unification still appear as very problematic.

Their differences are those of their respective relationships with the physical phenomena and "objects" they describe. The first one — the Theory of General Relativity — seems to maintain a direct relationship with the "objects" and phenomena aimed at, by the use of quantities defined on the space-time continuum. The second one — the Quantum Theory in its various dynamical extensions — although it provides also a thorough account of its relevant phenomenal domain, does it more indirectly: the quantities it deals with are specific — they imply *sui generic* relationships that are the core of the theory adequation — and need some kind of "translation", by definite and precise rules, to be put in correspondence with observable (classical) quantities in measurement apparatuses.[4] This need for such translation enlarges the distance between the theory and the experienced phenomena, but without altering the genuine theoretical status of Quantum theory, for it is a the Quantum Theory level that the fundamental representation stands, and that intelligibility is acquired and functions. It is not the observations that make the intelligibility of the quantum domain, it is the theoretical thought that orders the observations and transcribes them into its own terms, its concepts and relationships of concepts.

Nevertheless, it is due to this distance that — for psychological and historical reasons — Quantum Theory has long been and is still often qualified as a "mathematical formalism" — in contrast with a *mathematized physical theory.* The relation of the formalism with the actually observable physical quantities would require not only an "interpretation" of the mathematical quantities of the "formalism" — these quantities being the state vectors defined in Hilbert spaces and the linear operators, matrix or differential, acting on these states —, but an "interpretation" also of the whole theoretical, and even metatheoretical, array. It would be, in effect, another type of knowledge and the interpretation required would fundamentally be a philosophical one, about reality and knowledge. Negation of the former (reality) and reduction of the latter (knowledge) to observation, such was the claim of the "orthodox interpretation". One can however take an alternative position that looks more satisfactory, either from the physical as from the philosophical point of views (Paty 2003, 2009a). According to it, *Quantum Theory* is a *physical theory* in the proper meaning, whose theoretical quantities express *physical concepts*, for they give the mental and rational representation of the corresponding domain of physical reality — keeping this thought category in its broaded acception. The characteristic properties of the area, i.e. the specifically quantum phenomena, are in fact directly referred to the relations of concepts of the theoretical structure — as exemplified by the explanatory scope of the principle of linear superposition of the quantum state functions.

It happens that, despite the differences between the two quite different chosen theories — General Relativity and Quantum Theory —, the actual processes of invention of each of them did involve a kind of relation between the "mathematical formalism" and the physical conceptualization that was rather similar, as it will be shown now.

4. Einstein's general covariance

4.1. The invention of the General Theory of Relativity and the "drag of the physical thought by the mathematical forms"

We shall not enter in all the details of the way by which Einstein in trying to formulate a generally relativist Theory of Gravitation has been led to develop the General Theory of Relativity (see Pais 1982; Paty 1993, ch. 2–5). We want only to point at the rather unprecedented kind of relation that he established thereby between physical and mathematical thinkings.

Reflecting, shortly after having established the Special Theory of Relativity (Einstein 1905), on the scope of the principle of relativity in Physics, Einstein identified two states of affairs which, in his opinion, were problematic and limited the meaning and application of his theory. Note that this reflection concerned mainly Physics and not its mathematical formalization: it came to him in 1907 (Einstein 1907), three years before Einstein would take into account the space-time formalism as such — and even five years before he fully realized its importance and began to use it in his theoretical work.[5]

The first problem which he saw as a difficulty was the status of inertial motions which his theory considered to the exclusion of others. Einstein wondered why the principle of relativity — which was thought to rule all the laws of physics — would give privilege to inertial motions, uniform and rectilinear, which are such only once settled a referential point of view, arbitrary or anthropocentric, compared to most motions in the world that are general and varied. Nothing, physically, justifies this choice, if not our own peculiar situation with respect to such motions. (Let us note the ideas, underlying this remark, of non-referential objectivity and of totalizing or cosmological perspective; cf. Paty 2007.)

The second problematic aspect of the special theory was some unsufficiency it seemed to reveal as to the very purpose of the theory, which was giving privilege to the uniform inertial motions, when the first of the laws of dynamics, that of the falling bodies due to gravity, involved accelerated motions. However, for Einstein, the invariance of physical laws in inertial motion, which was strictly speaking the object of his theory, implied only kinematical considerations, those of redefining the concepts of space and time so as to put them in conformity with the requirement of that invariance. Dynamics was not directly concerned, contrarily to the conception which had led the parallel elaborations of Hendryk Lorentz and of Henri Poincaré, authors of a relativistic dynamics of the electron, and as well to the dynamical interpretation of Einstein's theory conceived by other physicists such as Max von Laue and mathematicians such as Hermann Minkowski.[6] Einstein, as for him, had considered dynamical theories only through their general required property of invariance under inertial motions: but this obliged him to reconcile the constancy of the speed of light independently of its source motion with the principle of relativity, and so to change the composition of velocities, and space and time themselves, and thus to reform the kinematics.

Einstein insisted on the kinematicald, and not dynamical character of the theory of relativity, which was an advantage — it had permitted to reformulate directly the concepts of space and time —, but also a disadvantage: space and time, now linked together, remained uncennected with the bodies and more generally with the matter they contained. On the whole, so he thought, they both had lost the absolute character imposed by Newtonian mechanics, as being redefined by the imposition of the principle of relativity, which provided them with some physical content, they still had something absolute, non-physical, as a space-time containing the bodies without being affected by them. To summarize this in one word, a theory of covariance alone without implying a dynamics was lacking something to be fully a physical theory: it was only a theoretical framework, in wait for more.

Precisely, the invocation of the falling bodies in uniform accelerated motion jointly with the gravitation field provided a clue to overcome this limitation. Einstein later spoke of this rapprochment as "the happiest thought of [his] life." It came to his mind as an "intuition" — bringing together in a synthetic view various properties — that he expressed by an imaged and "existential" formula: "If someone falls freely he no longer feels his own weight".[7] He transcribed it into a "thought experiment" — a theoretical summary made concrete in the thought as imaged phenomena —, that one of an elevator in free fall, and explicited it through a theoretical statement in terms of a "physical principle": the principle of equivalence between a uniform accelerated motion and a uniform gravitational field, reduced to, and justified by the identity of the inertial mass and the gravitational mass, admitted as a fact of general scope. Einstein thus realized that the stake was not so much to incorporate the gravitational field to the theory of relativity (for the inertial motions) than to use it as a means to overpass the privileged covariance of inertial motion towards that of motions of any kind.

He thus proposed, for the new theory he aimed at, a new object of a different nature from the previous one, as it combined a requirement of invariance (*covariance*, as it would be called), with a *dynamics*, that one of the gravitational field. Einstein expressed this change of perspective as follows: "The Special Theory of Relativity, which was nothing but the systematic development of the Electrodynamics of Maxwell and Lorentz, *was pointing however beyond its own limits*." (Einstein 1922; my emphasis.)

He formulated the problem in terms of general physical principles: the principle of relativity generalized to all kinds of motions whatsoever (i.e. a postulated *invariance* principle) and the principle of equivalence between the inertial and the gravitational masses (i.e. a *general fact* and a *dynamic principle*): their joint implementation would lead to formulate a (General) Relativistic Theory of the gravitational field.

Such was, so to speak, the platform of fundamental ideas from which Einstein embarked on the elaboration of the General Theory of Relativity. Note that these concepts and principles were essentially physical ones, in that they were directly referred to the phenomena of nature, and that their invocation practically did not call for mathematical formalization. Clearly this formalization is not absent, since the concepts or physical quantities involved and their relationships are mathematically expressed — metric invariant, formulas of transformation of space-time coordinates, composition of velocities, equations, etc. The formal expression of quantities serves, so to speak, and follows the physical thought, which orders their arrangement, by submitting them to the requirements (principles or laws) formulated with reference to phenomena. To the researcher's thinking, it is Physics that sets the pace — although mathematics, giving to concepts as quantities their exact expression, plays a role in the thinking of these quantities — for example, the thought of continuous quantities with their differential form. Note also that the movement that drives the thought — here physical thought —, if it is happens necessarily in a singular intelligence — marked by a personal style — is linked to the consciousness by the latter of some immanent necessity designated by a principle of objectivity, namely that covariance has a deep meaning only when it is a general one, independent of the particular referential point of views.

After this beginning, the "construction" of the theory — for it was indeed a construction, an intellectual one —, had to take another path, very different from that one, while remaining fundamentally in the same line of thought prepared in the way we have seen, guided by the concern for Physics. In his further work, Einstein had, in effect, to leave the more familiar ground of physical thought, where full attention is given to contents, for that, very distant at the beginning from the first, of the thought of forms and relationships independent of contents, and of the formal expressions that can be derived from them. This happened, however, only because Einstein had been able to express, in the first phase, the physical problem in terms of relationships of forms. His thought of forms, to which he would commit himself trustfully, did not forget the physical insight that had motivated it, even if he had to depart from it temporarily. It was therefore in a natural way that he met again the physical aspect of the problem, when ending his main journey that was the long elaborationt of General Relativity performed from 1907 to 1915.

He would indicate later on: "But the path has been more difficult than one might have expected, because it required to abandon Euclidean geometry" (cf. Einstein 1919). Einstein realized in 1912 that the problem as he had formulated it would find no solution with the Physics available and that, firstly, the physical bodies could not be compelled to comply with Euclidean geometry.[8] In terms of space-time, it was needed to leave the (pseudo)-Euclidean metrics, and to identify the gravitational potential with the metrics at each point. He was helped to think in those terms by his earlier work on the kinematics and his analysis of the physical meaning of space and time coordinates.

It appeared, as a matter of fact, that the obstacle was this physical meaning, linked with the Euclidean conception of space. Working with Minkowski's formulation of the 4-dimensions space-time, he dropped — provisionnally — the physical meaning of the space coordinates and distances, of the times and durations, and let freely go the mathematical reasoning, according to its proper way, without being any more limited by considering a physical content for these quantities. The distances had no more to be rigid and Euclidean, durations had no more to follow a "uniform flow" as Newton had defined them. A referential space (-time), whose structure would be left free of determinations could adapt itself to that one made-to-measure that would be given by the gravitational fields.

Einstein found in the theory of Gaussian surfaces the tool that enabled him to advance without the help of Euclidean metrics. He became familiar with the tensor calculus of Ricci and Levi-Civita and with the Geometry of Riemann, realizing henceforth the importance of the foundations of Geometry (Einstein 1922). He proceeded to develop mathematically the idea of general covariance, using the metric tensor to represent the gravitational field.

He obtained, in the end of 1915, the equations of General Relativity,[9] and with them he found the new physical meaning of the quantities, which carried, in their mathematical form, the non-Euclidean metrics given by the gravitational fields. Three consequences on the physical properties of bodies submitted to gravitational fields came out immediately, followed later on by many others: the explanation of the anomalous value of the secular advance of the perihelion of Mercury planet, the slowing down of clocks and the redshift of spectral lines.[10]

It is remarkable that an account of a formal nature — the mathematical expression of general covariance, any physical meaning of quantities being ignored — had the power to provide the fundamental piece that was lacking in the construction, namely the varying metrics at any four-dimension point. It was the free play of the mathematical relationships between the quantities involved that helped to get it: henceforth the physical meaning of the quantities modified in such a way was obtained. The temporary abandonment of any physical content had deleted the blocking cause, and allowed to establish new relationships required between the quantities by the formal condition, namely that the equations have the property of general covariance.[11] This was, Einstein wrote, "a purely mathematical task".

The blockage was due to the physical content previously attached to the quantities. The condition, first formal, of general covariance having been given preminence over the possible known physical content, all its relational implications could therefore be explored without restriction; it is this condition, explicited mathematically in that way, which provided then, through the new derived relationships, a new content to quantities: it was, truthly, their new physical content, if one takes into account that the condition of general covariance had been fundamentally formulated as a physical requirement for the relationships between the physical quantities.

One cannot forget, indeed, that this "formal" to which the thinking relied, had been designed and formulated to allow the full expression of the requirement of general covariance, but that this requirement was motivated by physical reasons — such was the starting consideration for the theory. In a sense, the formal point of view that Einstein met — and which would take henceforth an important place in both his Physics and his Epistemology — happened to fit exactly the program for Physics he had set forth — already in 1907 — and provided the key which opened the last doors to it.

Through his work of elaboration of the General Theory of Relativity, Einstein discovered that the formalism uncovers a really heuristic role: it became the indispensable tool not only for the expression of physical quantities — it had been always such through their mathematization — but also for the very discovery of the new quantities and of the laws that ruled them. Formalism had been instrumental in setting the problem and getting its solution, without having for that substituted the work of the conceptualization, a properly physical work, which had constituted the initial phase of his endeavor, and which he found again at the end of the route. For this physical concern had always been present, as he said himself, for example in 1920: "The mathematical form" of the theory "is only an instrument, and the most important is to consistently follow the thread of a few simple principles toward which the physical experience has led us . . . " (Einstein 1920). Such would be, from now on, his further constant attitude in his research on the Theories of Relativity and of the Unified Continuous Field.

Notwithstanding his insistance on Physics, Einstein did not underestimate for that the decisive importance of formal thought. Recalling, in 1946, in his Essay of intellectual autobiography, the way in which the General Theory of Relativity was established, he said:

> Equations of a complexity akin to those of the gravitational field can be obtained only through the discovery of a logically simple mathematical condition which determines completely or almost completely the equations. As soon as one has obtained such sufficiently strong formal conditions, one needs only to know a few facts to establish the theory. (Einstein 1946)

And he added: in the case of gravitation, what determines almost completely the equations, is "the four-dimensionality of space and its expression by a symmetric tensor, together with the invariance under the group of continuous transformations" (Einstein 1946).

4.2. The point of view of simplicity and the drag of physical thought by mathematical forms

The "point of view of simplicity" has permitted, in this case, to reach a formal construction intended toward the physical problem, of which, so to speak, it filters the link to the tissue of experience in providing a direct reading of the order underlying the phenomena. To do that, it takes a conceptual form, that one, in the case considered, of a geometrization of gravitation. This intellectual attitude was also that one he took in his research on the Unitary Field Theory, and he would also have liked to take it with regard to the Quantum Theory.[12] It was indeed a "point of view of simplicity", in that it reduced physical phenomena to their structural-principle aspect which admitted its transposition into an organizing mathematical condition.

What Einstein called the "point of view of simplicity" can also be seen as a "*drag of physical thought by mathematical forms*", that we can try to characterize first in a rather general way, as relative to the mathematical form of the physical concepts or quantities, and then in a more specific view in the light of the approach we just evoked about the conceptual leap that corresponds to the transition of the spacetime of Special Relativity to that of General Relativity.

About the first way, rather common since the mathematization of Mechanics and of the whole Physics from XVII^th^ to XIX^th^ centuries: when a "mathematical tool" is used to express a physical concept — for example, the differential form of the continuous quantities of classical physics —, this mathematical form is incorporated into the thinking of the concept, giving to it its relational properties. As a consequence Physics was gradually developed in its various areas through the analytical description — that of differential and integral calculus, and above all of the partial differential calculus — around the physical principles specific to each area. This effect of mathematical, analytical, formatting accompanied the physical conceptualization itself, through an agreement — an harmony according to intuition and intellection — between the form and the content, between the mathematical formalization and the thought of physical quantities as properly physical ones due to their linking to physical objects and phenomena. We can say in short that when the physical theory has been adequately formulated, the very form provides the content, which it includes, as it had been properly constituted to express this physical content.

The second way is less direct, since it implies a *separation*, at some point in the work of the thought, between, on the one side, the *mathematical form* that imposes new relationships in a direction dictated by the choice of the new physical principles and, on the other side, the *physical content* which faces limitations incompatible with the implications of this choice. The thought of the form — which can be justified physically, by the choice of the physical principles — becomes the means to overpass the thought of the too narrow physical contents: it is the way for the thought *to reach a new field of rationality*, needed to broaden the thinking of physics. The thought of the form is situated at the level of rational thought, in its intention as well as in its operation. It is seen afterwards, if it gets success, that the *thought of the form* has been fitted — by anticipation in the sense of the hypothetico-deductive manner — to the transformed physical thinking, and we are therefore founded to conclude that *it pertains the rational structures of the new physical thinking*. By this incorporation, the form has become physical as well — in the theoretical sense —, and it constitutes henceforth the proper expression of the new physical theory.

5. The thought of the physical specificity of quantum systems

Quantum physics presents a situation at first sight different from that of the domains of Physics that we just mentioned, due to the peculiarities it settled as to the relations between the theory and the phenomenas observed, peculiarities which appear to establish a greater distance than previously between form and content. We will not take again here the question of the various interpretations of Quantum Mechanics and, more generally, of the Quantum Theories based on it — Quantum Theory of Fields, etc. We shall propose first some remarks on the role of the formal and the physical in the effective thoughts of nowadays physicists of the area, showing that Quantum Theory, although it might appear as a "pure formalism", is the body of thought by which this area of physical nature becomes intelligible to them. Then we shall comment briefly on one of the creative moments of Quantum Theory, namely the work of Paul A. Dirac, in which the thought process is that of a drag of the physical thought by the mathematical form, as in the case of the invention of General Relativity, although with different modalities.

5.1. Remarks on the formal and the physical in the quantum domain

Although the Theory of Quantum Physics seemed to its founders and to their contemporaries, and often still seems to many, to differ from a Physical Theory in the usual sense of the Classical Physics and of the Theory of Relativity, the manner by which physicists practice it keeps a fundamental feature in common with the former: it is the *body (or instance) of intelligibility* of the physical systems and phenomena of the quantum domain. One has traditionally separated on the one hand the so-called "*mathematical formalism*" — that of the state functions defined in a Hilbert space and subject to the linear superposition principle, of Hermitian operators acting on these and commuting between them or not, which correspond to the dynamic variables of the systems, of the state equation whose solutions are the eigenfunctions and the associated eigenvalues, etc. —, which describes (indirectly) the physical system in an abstract and formal way, and, on the other hand, the result of *observation or measurement*, obtained in the form of numerical values of classical variables from the indication of the measument apparatus — that one being described by classical physics. The connection between the two is made possible, under the egid of the principle of correspondence, through the application of rules such as the probabilistic interpretation of the state function, the rule of projection or reduction by the apparatus of the state of the system on the measured eigenstate, etc.

In practice,[13] the determination of the properties of a quantum system from the results of measurements is obtained by taking the statistical data for the measured classical quantities, and by reconstructing the state function from the observed eigenfunctions, each one being given its relative amplitude through the Born rule ($a_i = |\psi_i|^2$),[14] which gives for the state function of the system: $\psi = \sum_i a_i \psi_i$. The operators, which are connected to the physical properties, and called traditionally "observables" although observation strictly speaking concerns only their eigenvalues —, are reconstituted from these. Today physicists conceive and represent, "intuitively", so to speak, the physical systems they deal with through these quantities and their *mode of use* — which may also mean: the type of their *relationships*. They associate to the formers the corresponding responses of the measurement apparatuses in terms of values of classical quantities, but these responses are, to their thought of the quantum phenomena, a kind of *auxiliary*, because the actual physical properties of the quantum systems are expressed at the *proper level of the quantum quantities* (see, for example, the effects arising from the *principle of superposition of the state function*, such as interferences, distant correlations of sub-systems, the intrication of interacting systems, etc.) (see Paty 2003). Therefore, the *quantum quantities* can be considered fully as *physical quantum concepts*, as they have the same function for the thought.

Physicists think quantum systems — atom, nucleus, particle, radiation — through their theoretical representation, so-called "formal", which is, in fact, their only way of thinking them, even when referring to the phenomena by which these systems come out. Everything happens in their thinking as if they had access to a kind of picture of the phenomena, algebraically rather than geometrically, through these dynamic variables as operators. They think the physical characteristics of quantum phenomena as being described by the theoretical magnitudes taken with their very mathematical form. Their thinking deals with the proper level — if any — of the quantum domain, as if they actually had access to it in this way, and indeed their representative imagination gives them this access through the "formalism"; it is therefore no exaggeration to qualify the so-called "formalism" as a "theory" in the proper meaning, a "quantum theory", understood as the structured organization of *quantum concepts*, expressed by the *quantum quantities* (the operators and the state functions).

In the present state of Quantum Physics, and whatever be the conceptions given of the "philosophical interpretation" of this type of knowledge, physical thought and formal thought go along together, the second being the means of the first. A new equilibrium has been established between the two among the physicists in this area, which does not prevent, however, some to have a more pronounced "physical sense" of phenomena than others, and others to be more inclined to deal with formal combinations (for example, towards the formulation of symmetries) which can possibly be meaningful in terms of physics, such as the development of gauge theories, for example, has shown (Paty 1988, ch. 7 and 8).

This reflection on the state of Quantum Physics in its current intellectual practice might be complemented by a further remark on the originating circumstances of its elaboration, where the mathematical expression of relations held to a vary large extent. Looking at how Quantum Physics has gradually established, from Planck's hypothesis of quantifying the energy exchanges between atom and radiation, to the elaboration of Quantum Mechanics and also to the further developments, one sees that it is through bringing to light — often indirectly — specific mathematical relationships between physical quantities, that the physical knowledge has been able to penetrate the atomic and subatomic world that escaped direct sensible perception. These relationships were found to correspond to physical features characteristic of quantum phenomena, and whose reason was actually to be looked deeper than in the combination of obvious, apparent, quantities; this deep exploration could only be done, from the intellectual, mental, point of view, by means of mathematics, which are eminently the science of relationships. To our retrospective view, Physics would formulate the significant quantities whose kind of (mathematical) relationships would permit to give account of the properties manifested in the phenomena — explaining and predicting them. It seems not unreasonable to think that it would thus give access to the quantum level of physical reality — but this would bring us back to the problems of philosophical interpretation on knowledge and nature.

Beyond the simple relationships of quantities at the quantum conceptual level, quantum theory itself can be seen in its sringing out and in its development as a case of drag of the physical thought by mathematical form. We come to an aspect of it now.

5.2. Dirac's conceptual and formal thought of quantum operators

The research that led to Wave Mechanics and to Quantum Mechanics in the years 1925-1927 have all been marked by the importance attached to the formalization by means of abstract quantities, whose mathematical behavior provided relationships and constraints which, referred to the experimental conditions, corresponded to the specific phenomena observed. Erwin Schrödinger with Wave Mechanics, as well as Max Born, Werner Heisenberg, Pascual Jordan and Paul A. Dirac with Quantum Mechanics, founded themselves on the Hamiltonian formulation of Classical Mechanics, which had shown its fertility, due to its generality, in all areas of Physics.[15]

Generally speaking, the path followed by the founders of Quantum Mechanics was, with different approaches for each of them, to let be guided by the indications of a "formal analogy" — in a structural sense — with the fundamental relations of classical mechanics in its Hamiltonian expression, more compact and general, but without keeping the same physical meaning nor the same peculiar mathematical form, for the dynamical variables of this formalism.

Dirac's researches are perhaps the most exemplary in this respect, particularly because of the interest of his thought of extending the concept of number to non-commutative magnitudes which allowed to treat various kinds of quantities, classical and quantum, "on a equal footing": such perspective is consonant with the idea of a proper representation of quantum systems, equally with that of classical systems. Geometry was important in Dirac'mind, either as space-time of the Theory of Relativity, either as spaces with more than three dimensions, either and above all as Projective Geometry for its intuitive call. In his work on Quantum Mechanics, Dirac tried, at least initially, to understand the algebraic relations — including non-commutation — in an intuitive way, inspired by geometry.

I will confine myself here to discuss an aspect of Dirac's research,[16] that one of his approach to non-commutative quantities as soon as they had been obtained by Heisenberg, from the calculation of transition amplitudes between the levels of the hydrogen atom. As it is known, Max Born recognized in them the mathematical properties of matrices: in the product of these quantities two by two, the order is of importance, $AB \neq BA$[17]. Dirac realized that the difference of Heisenberg products corresponded, except for a numerical factor ($i\hbar = ih/2\pi$), to Poisson brackets of Classical Mechanics, such as those involved in the Jacobi equation giving the variation with time of a dynamic variable F: $\frac{dF}{dt} = \{F, H\}$, with

$$\{F,H\} = \left\{ \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} \right\},$$

H being the Hamiltonian function.

He undertook to write the equations of Quantum Mechanics in the Hamiltonian formalism, by replacing, in the spirit of the correspondence principle, the Poisson brackets of Classical Mechanics by the differences of "Heisenberg products". He noted however that "the correspondence between the quantum and classical theories lies not so much in the limiting agreement when $\hbar \to 0$ as in the fact that the mathematical operations on the two theories obey in many cases the same laws" (Dirac 1925a, p. 649). From a simple agreement of numerical values in a passage to the limit between two different domains of theoretical validity, the correspondence principle acquired with this acception a heuristic status, that of the transport of a mathematical, structural, property from an area of physics into another one. The concern for numerical values — that is to say for the particular physical contents — gives way to that for form, which allows to cross the conceptual barrier and to explore the new area yet unformulated. Or, in other words: it is the form that matters, because it is what gives the expression of the law.

The equations written in this way had the same formal structure as the corresponding classical equations: to a product of two classical quantities corresponded a product of two quantum quantities. Dirac sought then how he had to transform the mathematical operations of the classical theory in such a way that the equations take the same form. The non-commutative multiplication led him to formulate the rules of a new calculus, that of a "quantum algebra" (Dirac 1925b). He then interpreted the quantities or magnitudes of this algebra as another kind of numbers, different from ordinary numbers (the "c-numbers", c for "classical" or "commuting"), and called them "q-numbers" (q for quantum).[18] Dirac was helped in these reflections by his ability to handle the Grassmann symbolic calculation used in Geometry. He formulated the differentiation of a dynamical quantum variable with respect to a parameter, in terms of algebraic operation, and found that the quantum variables, which appear in the fundamental equations of Quantum Mechanics considered in the Hamiltonien formalism were precisely the quantities x, y, etc. used by Heisenberg,[19] if one defined the commutator of the quantum magnitudes x and y by the relationship:

$$[x, y] = xy - yx = \frac{ih}{2\pi} \cdot \{x, y\}.$$

He obtained in this way the equation of time evolution for a variable $x(q, p)$, the fundamental equation of Quantum Mechanics:

$$\frac{dx}{dt} = \frac{ih}{2\pi}[x, H],$$

H being the Hamiltonian operator.

The essential difference between classical and quantum variables was the non commutation of the latter, the other characters of the two types of variables being the same: Dirac's consideration that the latter was another type of numbers allowed him to treat classical and quantum variables on an equal footing. It was to say later about his first ideas that we have just evoked: "I suppose that it was the main point in my early work, that I did appreciate that there would be a close analogy between the q-numbers and ordinary numbers".[20] This phrase actually summarizes well what was at stake, if we understand "close analogy" with the same meaning as the "formal analogies" invoked by Poincare and by William Thomson (Paty 2008).

An original feature of Dirac's work relatively to others who went along a neighbour direction was to have sought a common basis for the formulations of quantum and classical quantities, instead of contenting himself with the finding of a new formula. This feature is particularly significant about the nature of the physical thinking in its construction of a new area, because it enlightens the deep meaning of the work on formal properties, as a means to overcome the limitations of the known physical domain — hence the importance of the reference to some kind of "correspondence" between the old and the new theory. This overcoming is got, in the particular case, by basing oneself on a framework of theoretical thought that is strong enough to be taken for granted (the Hamiltonian formulation), and then by extending the meaning of a certain type of quantity in order to uncover the new territory. This extension is a new way of conceiving and representing quantities and involves, indeed, a new rationality of the thought of quantities.[21] At least, the path taken by the work of Dirac seems to indicate or suggest such a possibility.

However, the solutions of the equations of Quantum Mechanics obtained, in Dirac's theory scheme, by simple symbolic operations in terms of q-numbers, could not be confronted directly with experimental data. It was required, so it seemed, to infer, from the equations in q-numbers, the corresponding equations in c-numbers. In other, more familiar, words, it was required to pass from the quantum dynamical variables to the corresponding classical ones, loosing, so to speak, the benefit of the "leap of thought" that had been considered. This leads, from the point of view of the historical development of ideas on Quantum Mechanics, to the problem of interpretation. Dirac, although he had come *to think of quantum sytems with a symbolism appropriated to their meaning*, by extending the concept of numbers farther than numerical values or functions — and this can be seen also as an extension of meaning of *physical quantities* to those of the quantum formalism —, "on an equal footing", meaning an equal right to a physical meaning —, finally conformed himself to the operationalist "Copenhagen" interpretation.

It remains that the choice by Dirac, of the "formal analogy" for Quantum Mechanics through the Hamiltonian expression of the relationships between the dynamical variables, has been a very fortunate one, by its fertility when considering the results obtained, but also, we can understand it better nowadays, by its theoretical significance. It has helped to identify the most *specific characteristics* of quantum systems, the most "unthinkable" ones in any other theoretical and conceptual scheme. It has helped to *unveil* them, to *conceive* and to *think* them. The use of mathematical tools for thought — of a relatively unusual kind in this case — proved once again to be the privileged means of physics for its own overcoming (see Paty 2002, 2005).

The lesson of Dirac's thinking work, so we may estimate, implies effects for today. The theoretical representation of quantum systems by the state function and by the variables-operators of Quantum Mechanics can be seen as a physical representation in the most direct possible meaning, the word "direct" not excluding a high degree of abstraction. The "directness" of such a theoretical representation must be referred, as a matter of fact, to the greater *immediacy to the understanding*, whatever its degree of abstraction, rather than to its proximity to experimental results, the *physical character* being, as for it, referred to the *content of knowledge* of that representation relatively to phenomena (see Paty 2009b).

6. The lesson to be gained from creative reasoning

6.1. Invention through the guide of the form

As we have seen in what precedes, in both cases, the physicist who wants to formulate a satisfactory physical theory of a new given area of phenomena, initially met a barrier that seemed at first sight insuperable. This barrier was due to the limitation of validity inherent to the physical content of the concepts currently employed, taken from the previous theories, for which he hoped in a first phase that it would be enough to reorganize them inside the new theory under elaboration for the new domain.

Now, for each of these fundamental theories of matter, the path to the solution by crossing the barrier has been obtained as follows, in general terms: first, consider a guiding physical property for the significant quantities that might be considered fundamental, although not accounted for in the present physical theory, confront it with the available theory and concepts, and formulate what is to be expected (the problem) and what is demanding (the difficulty); second, forget the actual physical meaning of the concepts (or quantities), by emptying them of their physical admitted content (given by the previous theory or theories), regard them as mere mathematical quantities or variables devoid of a physical meaning in the usual sense; and finally, submit them to the formal operation (of transformation) thought of from the new properties in the physical domain under consideration in wait of their full theoretical description, being admitted that such operation actually expresses or translates (formally, mathematically) the condition corresponding to the required new specific properties (such operation having been stated in the first phase). Actually, the inadeqquacy or impossibility met with in the beginning was due to the incompatibility between the initial physical meaning of the quantities and the mathematical operation that corresponds to the new expected specific properties.

One has taken, in so doing, a part of the requirements of the physical problem, that one corresponding to the new expected property, this being only partial, as one deliberately omitted the physical content — the traditional previous one — of the quantities used. But one has also, as a matter of fact, extended the potentialities of expression of these quantities, by removing one (or more) of their restricting constraints — e.g., for General Relativity, the condition for space coordinates to refer to an euclidean space, or, in Quantum Mechanics, for numbers, to be non commutative). The "formal" operator acting on these quantities stripped of content turns them into new ones by providing them with another content (a mathematical one at this stage) given through the set of the new "formal relations" choosen. But, truly, the new content expressed by the relations generated by the "formal operator" is not merely a formal (mathematical) one, for it carries something of a physical "charge" (or meaning, or content), given from the new relations that proceed from a choice made for a physical purpose, that of the exigency that had been stated right at the start.

Such a choice was not a priori obvious and straightforward. It happened, in both the cases studied, that it was actually a good one, as it was fructiferous; it could have been different, and, indeed, other scientists have done different choices, or have qualified differently the problems. Science retains finally the choices which have been more efficient on the long term, and integrates them, at least partly, and history ratifies.[22] On the whole, there was between the known and the yet unknown a space of freedom for choice, for invention, for the operation of creative mind. This does not mean arbitrariness, unrationality, as one sees when trying to follow something of the actual process in the minds of the considered working scientists. As we have observed, rationality has a strong part in it, which is to be expected if the purpose for the mind is *to get intelligibility*. The search for intelligibility impulses the mind process and orients the transformation and formation of ideas. The freedom let for the mind in this process allows the *creation* of new *intelligible forms*, i.e. of forms which at the same time they are *formulated rationnally*, call with them *the conditions of their understanding* in the mind, this being nothing but the *form of rationality* needed for such a purpose. For the creation of new knowledges is most often associated, in the thinking work, with a widening of the forms of rationality which makes them possible (Paty 2005).

Let us take again the question of the *physical meaning* or *content* of the forms got by the quantities in the end of the process of thought as we have followed it: this physical content has been *blown into* the theoretical construction by *submitting it to the formal requirement* in which the considered physical peculiarity had been translated. The newly formulated quantities acquire their physical content or meaning from the new structured theory: under their new mathematical form they entail consequences that can be put in relation with the phenomena in the considered domain. They are submitted to experimental tests, which ensures whether they are approriate or not, that is to say, whether their *theoretically given physical meaning* does corresponds or not to the physical content given by the experiment.

6.2. The structures of rationality in the construction of intelligibility

We have examined a type of process of thinking that takes place in scientific invention or creation, in an area — Physics — which is indeed possibly more transparent to the analysis than others, for mathematical thought plays in it a fundamental role, although it was Physics — a science of Nature — that was considered, and mathematical thought reveals more directly than other ones its rational structuration. However the operation of mathematical thinking involves just as much as the other forms of scientific thinking (and more generally of symbolic thought) phases of creation and invention which can be characterized in rational terms, as they lead to set and understand something of the intelligible, but can in no way be reduced to purely logical operations. On the other hand, physical thought, even if it makes a privileged use of mathematics and of mathematical thought, is not to be confused with it. It does not invent (unless in exceptional, very specific, cases) new mathematical theories, and it contents itself simply with implementing those that are available. In fact, physical thought and mathematical thought do fertilize each other, as it is well known but exploring more this consideration would go beyond our purpose here.

Physical thought operates on physical concepts and physical theories, which refer to material phenomena of nature, and which are formulated through the use of quantities endowed with a mathematical form, and of relationships between these quantities. Physical thought focuses primarily on these quantities and on these relationships as they express physical contents, that is to say as they refer to physical phenomena and systems. The physical concepts and theories put in relation with the latter provide the intelligibility of them: such is their function for the mind. This exigency of intelligibility acts as the motor of physical thought (as well as of any scientific thought), particularly in its endeavors to fit the conceptual ant theoretical representation to the phenomena given in experience (often in an indirect manner). It is by following it in the mind process of the authors of pioneer works (insofar as it is feasible) that we may hope to get some element of answer to the question of how the *creation of new rational and objective knowledge is possible*.

In both mentioned cases, whose analysis has been outlined above, we have seen how the search for intelligibility by means of rationality leads the researcher to state his problem — which is a problem of objectivity formulated rationally — in conceptual and theoretical terms from the *point of view of physics*. The "formal" aspect plays a key role in both cases, and it would also play a specific role in other cases caught in the history of physics — for example, the thought of differential magnitudes, which helped to transform the science of mechanics in XVIII[th] century (see, for instance, Paty 2004). Here, however, the *mathematical form*, once established its reason of being from the physics perspective, plays a specific role of *dragging along* on the physical thought, since it will directly lead to make explicit the new physical conceptual contents: it is the very means of their construction. Under their new form, these are new physical concepts that so appear, deprived of continuity, in a certain way (from the point of view of the theoretical structure), with respect to the former ones. These concepts were not given initially, but they have not for that emerged from empirical data, they have been elaborated, *built*, *invented* in the mind, as the fruit of a "free decision" which will have permitted the qualitative jump between the old and the new concept. In this *invention*, the *intuition* that guided the choice (to which nothing obliged) remained situated in a *rational layout*, and it used a rational means to achieve its goal, this means being: let be led by the properties of the form, which correspond to the effect that was looked for.

To conclude, the conceptual and theoretical, physical, scientific thinking has had this ability to build by itself a new representation, that was present nowhere before, of the world of material phenomena. Physical thinking, here, have been helped in a privileged way by mathematical thinking as properly a thinking of the relationships between quantities. Holding these relationships farther than what the previous concepts allowed gave preeminence to the mathematical form, i.e. to the a priori rational over the "known physical" — or if one prefers, using a common but too schematic expression, for it ignores the role of conceptual thought, over the "empirical".

The proper role of mathematical thought in this work of the physical thought has retained our attention because it seems to reveal outstandingly the "structures of rationality" in this construction of intelligibility, which is itself a *creation* in the field of scientific representations. And so does mathematical formalization succeed in acting as a preferred means of rationalizing the world of physical phenomena.

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Resumo. Queremos considerar a questão, recorrente ao longo da história da filosofia, da relação entre o racional e a matemática, perguntando-nos em qual medida a estuturação do racional, que assegura a unidade da sua função sob formas diversificadas (e até sob uma

evolução de tais formas), poderia ser vista como homeomorfa à esta do pensamento matemático, tomado no seu movimento e concretizado nas suas teorias. Essa ideia, na verdade tão antiga quanto a própria filosofia, apesar de não ter sido dominante, encontra-se sempre até hoje em um certo grau no pensamento da ciência moderna, em Descartes bem como em Kant, Poincaré ou Einstein (e alguns outros cientistas e filósofos). Ela foi muitas vezes duramente criticada, notavelmente no período contemporâneo, seja em decorrência do fracasso do programa logístico, seja devido à diversidade dos conhecimentos "empíricos", seja, de maneira geral, em razão do carater transitório, evolutivo e construído dos conhecimentos científicos. No entanto, o exame do pensamento científico no seu procedimento inventivo e criador permite caracterizar este pensamento como sendo uma forma racional, cujas configurações podem ser detalhadas com certa precisão. No presente trabalho esboçamos algumas exigências filosóficas para tal programa de pesquisa, dentro das quais uma harmonização, e até uma conciliação, das noções de racional, de visão intuitiva e de pensamento criador. Em seguida, examinaremos alguns procedimentos de pensamento científico criador a respeito do conhecimento e da compreensão do mundo, diferentes desses da matemática e no entanto ficando em estreito relacionamento com esta. As teorias físicas contemporâneas são testemunhas privilegiadas a esse respeito, pois nelas o pensamento racional dos fenômenos recorre, de maneira intrínseca, ao pensamento matemático, este contribuindo à estruturação e à expressão dos seus conceitos (expressão que implica os conteúdos físicos desses mesmos conceitos). A Teoria da Relatividade Geral e a Teoria Quântica são exemplares a esse respeito, pois elas revelam diretamente o que pode ser chamado de "arrastamento do pensamento físico pela forma matemática": este último permite ultrapassar os limites do conhecimento anterior. Tal procedimento está estreitamente ligado às modalidades e à estrutura do pensamento rational subjacente. É isso que tentaremos mostrar.

Palavras-chave: Conhecimento científico; estruturação do racional; forma matemática; pensamento científico; pensamento criador; pensamento matemático; pensamento físico; programa fundacional logístico; visão intuitiva; relatividade Geral; teoria quântica.

Notes

[1] I borrow the expression "formal contents" from Gilles G. Granger (Granger 1994). On progress in mathematics, see Cavaillès 1947, 1962.

[2] For an analysis in this sense of the debates on this subject about Geometry and Experience in the context of the Theory of Relativity, see Paty 1993, chapters 6 and 7.

[3] With respect to the quantum domain, this distance can be symbolized by Avogadro number, 6.10^{23} molecules per gram-molecule.

[4] I try to formulate this difference in general terms that avoid the "orthodox" or "operationalist" interpretation flavour. See Paty 2009b.

[5] Respectively, Einstein 1910, 1912. See Pais 1982, Paty 2003, chapter 4.

[6] Paty 1993, chapters 2 and 3, 1996.

[7] He did it in retrospective evocations: Einstein 1922, 1955.

[8] The problem of the deformation of a rigid bar driven in a rotation motion had a great influence on his thinking: he gave it a kinematic solution (deformation of the reference

space, given by the physical or practical Geometry), and not a dynamical one through the action of forces, which on the contrary was preferred by the other physicists.

9 Einstein 1915. The equation is: $R^{\mu\nu} = -\kappa(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T)$, or $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\kappa T^{\mu\nu}$. In the second form of the equation, the first member represents the field quantities (the 10 components $g^{\mu\nu}$); $R^{\mu\nu}$ is the Ricci tensor, and R the scalar curvature. The second member contains the energy-momentum tensor ($T^{\mu\nu}$), which plays the role of the field source.

10 The General Theory of Relativity is exposed as a whole in the extended article (also published as a book) Einstein 1916. The prediction of gravitational waves is given in a further work (Einstein 1918), and the first scientific theory of Cosmology in Einstein 1917. About gravitational waves, one must recall that Poincaré was pionneer in predicting their existence, in 1905, as a consequence of his proposal to modify the Newtonian Theory of Gravitation on the basis of the relativist (in the restricted, inertial, sense) invariance obtained from his relativistic electrodynamics (instead of being instantaneous, gravitational attraction was propagated with the finite velocity of electromagnetic waves, c): he called them "gravitic waves" (Poincaré 1905). The word "gravitational waves" was universally admitted later on with Einstein's General Theory of Relativity.

11 Equations called "generally covariant". Einstein wrote, in the Introduction to his 1916 big mémoire: "The postulate of general relativity leads to the requirement that the equations of Physics be covariant under general transformations of the coordinates x_1, x_2, x_3 and x_4" (Einstein 1916, part A, introduction).

12 On Einstein's program towards a theory for Quantum Physics, see Paty 1996, to be published.

13 The actual procedure of physicists, which we try to sum up here, is independent of the philosophical interpretation of the type proposed by Niels Bohr, and even opposed to it, if we consider (see further down in this paragraph and in the following one), that classical quantities are *auxiliary* and that the physical thought is functioning at the "level of quantum magnitudes" or concepts. The "orthodox" interpretation considers that the only physical level accessible by the senses as well as by the thought is the classical level. For a discussion of these views, see Paty 2009a and b.

14 a_i is the "probability amplitude" of the eigenstate ψ_i.

15 Schrödinger 1926; Heisenberg 1925; Born & Jordan 1925; Born, Heisenberg & Jordan 1926; Born 1926, 1927; Dirac 1925 a, b, 1926 a, b, c, d, 1930. See: Jammer 1966; Mehra & Rechenberg 1982; Kragh 1982; Darrigol 1992; Bitbol & Darrigol 1993; Paty 1993b.

16 On Dirac's approach to Quantum Mecanics, in connection with the contemporary works, see especally Darrigol 1992.

17 Heisenberg [1925], Born [1926].

18 Dirac 1926a, p. 562. See also Dirac 1926b.

19 Dirac 1925b, p. 647–8.

20 Dirac, as quoted in Mehra & Rechenberg 1982, vol. 4, p. 162–3. These authors indicate the inspiration received by Dirac from Projective Geometry in the expression of the mathematical laws of the q-numbers.

21 On this question, see Paty 2001c.

22 It retains "science as judged", as Gaston Bachelard expressed it.