doi: 10.5007/1808-1711.2013v17n2p217

Putnam and the Indispensability of Mathematics

Otávio Bueno

Abstract. In this paper, I examine Putnam's nuanced views in the philosophy of mathematics, distinguishing three proposals: modalism (an interpretation of mathematics in terms of modal logic), quasi-empirical realism (that emphasizes the role and use of quasi-empirical methods in mathematics), and an indispensability view (that highlights the indispensable role of quantification over mathematical objects and the support such quantification provides for a realist interpretation of mathematics). I argue that, as he shifted through these views, Putnam aimed to preserve a semantic realist account of mathematics that avoids platonism. In the end, however, each of the proposals faces significant difficulties. A form of skepticism then emerges.

Keywords: Putnam; indispensability argument; modalism; philosophy of mathematics; set; Quine.

1. Introduction

Over several decades, Hilary Putnam has developed a series of nuanced and influential views in the philosophy of mathematics. He initially developed a modal interpretation of mathematics: an interpretation of mathematics in terms of modal logic (Putnam 1967[1979]). He then highlighted the significance of quasi-empirical methods for the proper interpretation of mathematics (Putnam 1975[1979]). Finally, he articulated a version of the indispensability of mathematics, influenced by Quine's formulation, but importantly different from Quine's (Putnam 1971[1979]), and became increasingly critic of all the main approaches in the philosophy of mathematics (Putnam 1979[1994]). More recently, he reassessed the indispensability argument and its proper interpretation (Putnam 2012a).

In this paper, I examine some of the shifts that Putnam's position has had, but also its continuity. Throughout Putnam's reflections on the philosophy of mathematics, we find a systematic attempt to provide a realist interpretation of mathematics and to resist platonism. I argue that, as opposed to Quine's, Putnam's view does not support a platonist stance, but only a form of semantic realism about mathematics. Central to his view is the indispensability of mathematics, but one needs to be careful about how to interpret properly this feature, since it need not lead to platonism—even if the so-called indispensability argument went through. In the end, given the ultimate failure of this argument, and the significant problems faced by all the views examined, I end on a skeptical note.

2. Putnam$_1$: Modalism

Putnam has developed at least three different views in the philosophy of mathematics: modalism, quasi-empirical realism, and an indispensability view. I'll consider each of them in turn.

Modalism (or mathematics as modal logic) is the view according to which mathematical statements can be interpreted as modal statements about the possibility of certain structures (Putnam 1967[1979], p.43–59; Putnam 1979[1994], p.507–508).[1] Putnam$_1$ advances this view. Consider, for instance, a statement to the effect that there are infinitely many prime numbers. This statement can be interpreted in terms of two modal statements (in a modal second-order language):

(1) If there were structures satisfying the axioms of Peano arithmetic, it would be true in those structures that there are infinitely many prime numbers.

(2) It is possible that there are structures satisfying the axioms of Peano arithmetic.

(Statement (2) is needed in order to prevent that (1) be vacuously satisfied, in which case any translated statement would end up being taken as true.) In this way, no commitment to the existence of mathematical objects emerges, since only the possibility of certain structures is entertained. In particular, quantification over mathematical objects is only done within the scope of modal operators. What is offered here is a systematic translation of mathematical statements into modal claims. Since this translation can be applied systematically to every line in a proof of a theorem, proofs and what they establish are, thus, also preserved.

Formulated in this way, modalism about mathematics has three significant features: (i) The *objectivity* of mathematics can be secured without the commitment to mathematical objects. Once certain mathematical concepts are introduced, and a given logic is adopted, it is not up to us to decide which results hold and which do not. This ultimately depends on the particular concepts that are introduced and the relations among them. (ii) A *realist* view of mathematics can be maintained, in the sense that mathematical statements have truth-values, and bivalence is preserved. This is, of course, semantic realism, and it is opposed to instrumentalism (according to which mathematical statements do not have truth-values) and to intuitionism (according to which, for at least some mathematical statements, it may not be determined whether they do have a truth-value or not). (iii) A *platonist* view of mathematics is *resisted*. According to platonism, mathematical objects exist, are causally inert, and are not spatiotemporally located. Modalism does not involve any commitment to such objects: quantification over them is only invoked within the scope of modal operators, and only the possibility of certain structures is asserted.

But what kind of objectivity is, in fact, preserved in this account?[2] There are at least two kinds: (a) According to one kind, something is objective if it doesn't depend on us whether a certain result holds or not. Objectivity, in this sense, allows for a plurality of different (but perfectly objective) solutions to a given problem. For example, it is an objective fact about Zermelo–Fraenkel set theory with the axiom of choice (ZFC) that it is compatible with many different values for the continuum hypothesis. Each of these values provides an objective fact about the sets in question.

(b) Another, and stronger, sense of objectivity insists that something is objective as long as when there is disagreement about it, at best only one of the parties in the dispute is right (and perhaps none). Objectivity, in this sense, requires uniqueness in the solution of the problem under consideration. If we consider debates over the continuum hypothesis, depending on the set-theoretic model one adopts, more than one answer to this issue can be adopted.This suggests that, in the second sense of objectivity, the issue of the continuum hypothesis would not be objective, given the plurality of allowed solutions. This is, of course, a significant contrast with the first sense of objectivity, according to which answers to the continuum hypothesis are indeed objective.

It's not clear which of these senses of objectivity (if any) Putnam is in fact invoking. To be on the safe side, I will tentatively adopt the first, weaker sense as the relevant one, since it is robust enough to make mathematics importantly objective.

Despite these attractive features, modalism faces several difficulties. I will consider two of them. (a) In order to secure the objectivity of mathematics, modalism requires the commitment to a primitive modal notion. However, what grounds the possibility of the relevant structures? More specifically, if we say that it's possible that there are set-theoretic structures satisfying ZFC axioms, exactly what does ground such a claim? It's implausible to maintain that there are inaccessibly many concrete objects (say, inaccessibly many mereological atoms) to support this possibility claim, since we have no reason to believe that there are that many concrete things. Moreover, the concept of *inaccessibly many* is a fundamentally set-theoretic concept. We would then be offering an interpretation of set theory that ultimately presupposes that set-theoretic concepts have already been properly formulated. This seems to vitiate the approach, since the adequacy of the interpretation now rests on the adequacy of the original formulation of the relevant set-theoretic concepts. If the latter founders, so does the former.

Suppose, however, that there were enough concrete (spatiotemporally located) entities to support the claim that it is possible that there are set-theoretic structures with inaccessibly many sets. In this case, there would be no need to introduce the modal operators, since the relevant set-theoretic structures could all be interpreted directly using the objects in question, by systematically assigning sets to the concrete entities. This would provide the benefits of modalism (objectivity, semantic realism, and non-platonism) without the need to invoke primitive modal operators. In the end, modalism would be effectively abandoned.

Perhaps one could invoke Lewisian possible worlds (Lewis 1986) to ground the possibility of the relevant structures. So when it is asserted that it is possible that there are structures satisfying the axioms of Peano arithmetic, this means that there is a concrete world, inaccessible from ours, in which it is true that there are structures satisfying the axioms of Peano arithmetic. This suggestion, however, would be similarly problematic. Even though these worlds are spatiotemporally located, we have no causal access to them. Thus reference to such worlds is at least as problematic as reference to mathematical objects. After all, both worlds and mathematical entities are causally inaccessible and thus require a special referential apparatus. Simply stating that we refer to independently existing mathematical objects or possible worlds just by describing their properties begs the question. The problem at hand is precisely to know how we can secure such reference in the first place. In any case, even if reference to both kinds of objects could be easily secured (as platonists sometimes claim about mathematical objects), this feature would be of no help for modalists, given their non-platonist stance. If the goal is to avoid platonism, invoking Lewisian possible worlds will be of no help here.

There are, of course, many additional views in the metaphysics of modality, and I am not suggesting that these considerations make modalism completely unworkable. But the concerns here suggest that without a proper account of what grounds the possibility of the relevant structures, it is unclear exactly how modalists can maintain a realist view of mathematics. The onus of proof lies now with them to establish properly their view.

(b) As we saw, modalism aims to provide a general strategy to avoid commitment to mathematical objects, and thus it tries to resist platonism about mathematics. Even if this strategy basically works, it ends up going too far. If the modal translations were applied to the mathematics used in science, we would lose the empirical content of scientific theories. For in this case, we would only assert the possibility of certain structures rather than that certain physical configurations in fact obtain (see Field 1989, p.252–68).

In response, perhaps an actuality operator, @, could be introduced to avoid this difficulty. Reference to actual physical structures could be secured by prefixing them by the actuality operator @, thus guaranteeing the preservation of the physical content of the scientific theories in question (for an implementation of this strategy, see Friedman 2005).

The problem with this suggestion, however, is that it presupposes that we can draw a sharp distinction between nominalistic and mathematical content, that is, a distinction between the content that does not involve any reference to mathematical objects (nominalistic content) and the content that does involve such reference (mathematical content). But it is unclear that such a distinction can in fact be drawn without implementing a Field-type nominalization of mathematics, which aims precisely to separate the nominalist content of a scientific theory from its mathematical content (Field 1980 and 1989; see also Azzouni 2011). There are, however, serious concerns about the viability of this form of nominalism about mathematics (for a survey and references, see Bueno 2013 and Section 4 below). Thus, introducing the actuality operator does not help in the end.[3]

3. Putnam$_2$: Quasi-empirical Realism

The second main philosophical interpretation of mathematics that Putnam advances (Putnam$_2$) is *quasi-empirical realism*. The central idea is that something *analogous* to empirical reasoning is found in mathematics (Putnam 1975[1979], p.60–78, and 1979[1994], p.504–7). This kind of reasoning is invoked in the choice of mathematical axioms whenever support for an axiom is provided by the consequences that can be derived from it. This kind of reasoning is also involved when types of plausible reasoning that do not amount to full proof are employed in mathematics—a kind of reasoning George Pólya examined in great detail (Pólya 1954 and 1962).

In addition to these considerations, a methodological similarity between mathematics and the empirical sciences is highlighted: both fields involve choice among theories based on theoretical virtues or criteria. Among the familiar Quinean criteria, two are highlighted: *conservatism* (the attempt to preserve principles that have been widely accepted in a given domain), and *simplicity* (the preference for simpler theories). These are, of course, familiar constraints in scientific practice, and for the quasi-empirical realist, they also play a significant role in mathematics, particularly in the choice of new axioms. In some cases, however, such criteria produce conflicting rankings. For instance, the preservation of some principles may entail informational loss in light of a less conservative change. Alternatively, a simpler system of principles may be less informative too. Similarly to what happens in science, in the case of such conflicts, different weights are assigned to the various criteria so that hopefully a well-motivated choice can be made.

There is one additional trait that the quasi-empiricist realist adds to these familiar criteria, and it is crucial for the choice of mathematical principles: agreement with mathematical "intuitions". One can think of such intuitions as providing data that ought to be preserved or information about the relevant objects that needs to be accommodated. Kurt Gödel has famously defended a view about mathematical intuition in which such intuition is understood as a form of "perception" of mathematical objects (see Gödel 1964). Putnam, however, wants to resist this account, which he thinks introduces a mysterious component into the discussion:

> On my view, mathematical "intuitions" are not mysterious "perceptions" of mathematical objects [as opposed to what happens in Gödel's account], nor do they have a single source (Putnam 1979[1994], p.506).

One may disagree with Gödel's account of the "perception" of mathematical objects. But it seems hasty to consider this kind of perception "mysterious". Consider, first, the ordinary perception of a physical object. It has three important features, which contribute crucially to make perception epistemologically significant:

(i) Perception is *factive*: If one perceives that *P*, then *P*. Given this condition, perception can be used as a source of reliable information about the world and, ultimately, of knowledge.

(ii) Perception has a distinctive *cognitive phenomenology*: There is something that is like to perceive an object. In virtue of such phenomenology, features of the objects of perception can be determined and studied perceptually (e.g. by tracking changes in their properties over time).

(iii) Perception is *robust*: What one perceives does not depend on one's beliefs (although the interpretation of what is perceived does). Perception ultimately depends on the scene before one's eyes and the resulting perceptual experiences. Moreover, perception is resilient to changes of belief, as some cases of perceptual illusions illustrate. Although we may know that two parallel lines, with arrows oriented in opposite directions at the lines' edges, have the same length, they still seem to be of different lengths. In this respect, perception is invariant under changes of belief.

It is important to note that a Gödelian account of the "perception" of mathematical objects, which can be called *mathematical intuition*, has exactly the same features:

(i) Intuition is *factive*: If one has the intuition of a mathematical fact *P*, then *P*. Similarly to the case of ordinary perception, without this condition in place, an intuition of a mathematical fact could not be a source of reliable information about the objects in question nor, ultimately, a source of knowledge.

(ii) Intuition has a distinctive *cognitive phenomenology*: There is something that is like to have a mathematical intuition. Certain axioms, Gödel tells us, present themselves to us as properly characterizing the situation they are about. They specify properties and relations among the relevant objects that seem to hold given the concepts that are invoked. Consider, for instance, the axiom of extensionality in set theory. If it is part of the concept of *set* that a set's individuality is determined by the members that set has, then it is not possible for two distinct sets to have the same members, nor is it possible for the same set to have different members. Given the concept of *set*, as something individuated by its members, the axiom of extensionality seems to be true. As Gödel points out, the intuition of set-theoretic objects can be "seen from the fact that the axioms [of set theory] force themselves upon us as being true" (Gödel 1964, p.485).

(iii) Intuition is *robust*: The mathematical intuitions one has do not depend on one's beliefs about the objects under consideration. They depend on the features of the relevant concepts and how they categorize the objects and relations in question. Given the concepts, mathematical intuition allows one to apprehend mathematical objects as having certain properties and lacking others. These features of intuition are invariant under changes in belief about the relevant mathematical objects, since the intuitions emerge from the features of the concepts in question rather than the beliefs one may have about the objects. However, the intuitions are not invariant under changes in the concepts, since it is via the concepts that the intuitions obtain.

Given that perception and mathematical intuition share these crucial features, it doesn't seem that the latter should be mysterious if the former isn't.[4]

Having said that, I don't want to suggest that a Gödelian account of mathematical intuition is problem-free. For instance, nominalists will resist the object-oriented nature of the account, that is, the fact that the specification of the concepts is taken to display features of (previously existing) objects. How could it be guaranteed that mathematical objects that exist independently of our specifications are correctly picked out by our concepts? This problem does not arise for nominalists, since they do not assume that mathematical objects exist (see Field 1980, Hellman 1989, and Azzouni 2004). Similarly, this problem also does not arise for those who are agnostic about the independent existence of mathematical objects, since they do not assume that mathematical concepts need to pick out independently existing mathematical objects (Bueno 2008). The mathematical agnostic can obtain verbal agreement with the platonist by noting that it is unproblematic to talk about the existence of mathematical objects once they are specified by suitable principles: after all, these objects are not taken to exist independently of our ways of characterizing them. In this respect, postulation of mathematical objects is very similar to the postulation of fictional objects. Given certain mathematical concepts, or concepts of fictional characters, certain objects are determined. But they need not exist independently of the relevant characterizations (see also Bueno 2009).

It can also be questioned to what extent the account of mathematical intuition just sketched is indeed robust. Some intuitions about sets may depend on beliefs about which sets one is considering and which properties these sets should exhibit. Consider the debates over the axiom of choice. After Zermelo explicitly formulated the axiom in 1904, many mathematicians resisted it, even some who had previously used the axiom in their own work (in some formulation) without realizing it.[5] Perhaps the beliefs these mathematicians had about sets changed their attitude toward the acceptability of what they apprehended via mathematical intuition, which eventually led them to question the axiom of choice (despite having implicitly used it before). This seems to question the robustness of mathematical intuition, given that beliefs seem to interfere with the content of what is apprehended and the corresponding assignment of truth-value to the relevant axiom.

It may be argued that there is an important difference between perception of physical objects and mathematical intuition. We understand pretty well how perception works; there is a clear mechanism involving the functioning of the eyes and the brain, as well as the behavior of light that accounts for how we perceive the objects that we do, and why we can't perceive the objects that we don't. In contrast, there is no such account for mathematical intuition, since it is not entirely clear in virtue of what mathematical intuition obtains, and why it fails when it does. We know under what conditions we can rely on perception and under what conditions we can't (for instance, if we had too much to drink, we may not trust entirely the perceptual experiences we have). In contrast, it is not so clear under what conditions mathematical intuitions may not be followed (although drinking too much is very likely to have an impact on mathematical intuitions too!). This questions one aspect of the similarity between perception and mathematical intuition. Interestingly, it lends some support for Putnam's point about the mysteriousness of mathematical intuition: although perception relies on a clearly understood mechanism, the situation with mathematical intuition seems to be more fragile.[6]

Perhaps this particular difference is what Putnam had in mind, in which case there would indeed be an important distinction between perceptions and mathematical intuitions. But since this point is not explicitly made in Putnam's paper (1979[1994]), it is difficult to assess it. In any case, it may be argued that, properly developed and taking into account Husserl's phenomenology, there is an account of mathematical intuition that is importantly analogous to perception, and it includes a certain mechanism of conceptual refinement that is significant to the proper conceptualization of the objects of mathematical intuition (see Tieszen 1989 and 2011, Parsons 2008, and Chudnoff 2013). Perhaps this would help to address the worry about the mysteriousness of mathematical intuition—although far more needs to be said about this issue.[7]

Putnam does indicate some additional traits involved in the account of mathematical intuition that informs quasi-empirical realism. He is particularly concerned with intuitions of the axioms of set theory, and highlights an account, emerging from Quine, that (if properly implemented) does have the virtue of making the "self-evidence" of the comprehension axioms of set theory fairly straightforward. The account is grounded in mundane linguistic habits of avoiding repetition of predicate expressions. Putnam notes:

> The Mill–Wittgenstein story—that mathematical induction [...] started out as a Baconian induction and was elevated to a different status along the line [eventually becoming something analytic]—seems right for mathematical

> induction, but not for set theory. Quine himself gives a plausible account of the origin of the "self evidence" of the comprehension axioms of set theory (these say that every condition determines a set, if we ignore the problem of avoiding the Russell Paradox). In *Ontological Relativity and Other Essays*, Quine points out that quantification over predicate letters occurs in natural language quite unconsciously as a mere device for avoiding awkward repetition of whole predicate expressions (Putnam 1979[1994], p.506).

It is curious that while Putnam acknowledges the self-evidence of the comprehension axioms of set theory, he dismisses so easily the set-theoretic paradoxes. *Prima facie*, the existence of such paradoxes seems to undermine any such self-evidence: it poses a significant issue to any account of mathematical intuition that supports the intuitive nature of these axioms.[8]

Putnam then continues:

> In effect, the use of what Quine calls 'virtual classes' (that is, classes which can easily be *eliminated* from discourse) leads automatically to quantification over predicates [...]; and quantification of predicates leads to precisely one of Cantor's two notions of a set: *the extension of a predicate*. The fact that the origin of the idea that every condition determines a set may have been something as mundane as everyday linguistic habits of avoiding the repetition of long expressions does not mean that the existence of sets must be questioned even after we have erected a successful theory (which, we hope, avoids the paradoxes) (Putnam 1979[1994], p.506).

But what exactly is the account of mathematical intuition suggested in these passages? Even if Quine's strategy to accommodate the "self-evidence" of the comprehension axioms of set theory in terms of everyday linguistic habits worked, it's not clear how this strategy could be generalized to other axioms. The axiom of extensionality could have been an even better candidate for something that is "self-evident". But it is unclear which everyday linguistic habit this axiom encodes. And even if it did encode some such linguistic habit, one still needs to account for the intuition underlying the axiom, namely, that only the members of a set are responsible for the set's identity. This trait seems to go well beyond just some linguistic habits. Thus, far more needs to be said to explain such intuition based on linguistic habits alone.

Moreover, how robust is the suggested account of mathematical intuition? Suppose we need to form a set out of objects that lack well-defined identity conditions, such as certain quantum particles according to some interpretations of non-relativist quantum mechanics. In this case, the axiom of extensionality needs to be restricted, since the members of the set in question lack a requirement for the application of the axiom: well-defined identity conditions. If the members lack such conditions, the resulting sets cannot be formed, since the identity of such sets depends on the identity of their members, which are lacking. As a result, it is not surprising that the extensionality axiom has been constrained in the context of set theories developed to accommodate the foundations of quantum mechanics: quasi-set theories (see French and Krause 2006). It is also questionable whether the relevant mathematical intuition about extensionality is robust, since it seems to depend on beliefs about the properties of the objects under consideration (quantum particles under a certain interpretation of non-relativist quantum mechanics).

Finally, suppose there are conflicting mathematical intuitions about a given axiom, such as the axiom of choice. How can such a conflict be resolved? Typically, the resolution (to the extent that there is one) is not implemented via such intuitions, but by examining the consequences provided by the axiom in question and comparing the resulting consequences obtained by the negation of the axiom. The fact that a plurality of non-equivalent set theories has been developed—some with the axiom of choice, some without it—suggests that, to the extent that mathematical intuitions play any role in mathematical practice, they do not guarantee a unique outcome, and for the reasons discussed above, may not be robust. (Of course, this is also an issue for Gödel's own account. But, on his view, there is only one correct understanding of *set*. The difficulty, for Gödel, consists in accommodating the plurality of set theories given the monist view he favored.)

The point here is that when there is disagreement about the outcome of certain mathematical intuitions—some favor, while others question a given axiom—a way of resolving the disagreement is to invoke the quasi-empirical realist's criteria for theory, or axiom, choice in mathematics (simplicity, conservatism, consequences obtained). The problem is that one of the motivations for the introduction of mathematical intuition was to provide a way out when the criteria for axiom choice generated conflicting outcomes. The quasi-empirical realist may end up entangled in a decision conundrum, using mathematical intuition to select axioms when there are conflicting recommendations from theoretical criteria, and using theoretical criteria to select axioms when there are conflicting recommendations from mathematical intuition. As a result, in these cases, the quasi-empirical realist may end up trapped in the tension among the various components of the view.

4. Putnam$_3$: Indispensability View

The third view Putnam developed (Putnam$_3$) emphasized the indispensable role played by mathematics in science. This view generated a substantial discussion surrounding the so-called indispensability argument. According to Mark Colyvan, the Quine–Putnam indispensability argument can be formulated as follows (Colyvan 2001, p.11):

(P_1) We ought to have ontological commitment to all and only the objects that are indispensable to our best scientific theories.

(P_2) Mathematical objects are indispensable to our best scientific theories [in the sense that reference to such objects is indispensable to the theories in question].
Therefore, we ought to have ontological commitment to mathematical objects.

Formulated in this way, this is an argument for platonism about mathematics (according to which mathematical objects exist). This may be a version of Quine's indispensability argument, but it is not a characterization of Putnam's.[9]

For Putnam, the indispensability of mathematics for physics provides evidence that mathematics is true under a realist interpretation (Putnam 1975[1979], p.60–78, 1971[1979], p.323–57, and Putnam 2012a, p.181–201). Given the modalist approach to mathematics (Putnam_1), and as opposed to Quine's version of the indispensability argument, indispensability considerations provide no support for platonism, that is, for the existence of mathematical objects (Putnam 2012a, p.182–183). For Putnam, the indispensability argument supports the objectivity of mathematics. According to him:

> My "indispensability" argument [as opposed to Quine's] was an argument for the objectivity of mathematics in a realist sense—that is, for the idea that mathematical truth must not be identified with provability. Quine's indispensability argument was an argument for "reluctant Platonism", which he himself characterized as accepting the existence of "intangible objects" (numbers and sets) (Putnam 2012a, p.183).

The target of Putnam's indispensability argument is the intuitionist who identifies mathematical truth with provability. The target of Quine's indispensability argument is the scientific realist who tries to avoid the commitment to mathematical objects. (Putnam agrees this is not a happy combination of views.) This is, in fact, Quine's own view: a reluctant platonist.

In fairness to Colyvan, there are passages in Putnam's early work that suggest a commitment to mathematical objects. For instance, Putnam notes:

> So far I have been developing an argument for realism along roughly the following lines: quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes (Putnam 1971[1979], p.347).

In the discussion following this passage, Putnam considers—and rejects—a number of responses to the indispensability argument, particularly those advanced by a fictionalist, who insists that despite quantifying over mathematical objects, one can resist the commitment to their existence (1971[1979], p.347–56). Given the rejection of fictionalism, it is understandable that Colyvan would attribute to Putnam the kind of indispensability argument that ultimately belongs to Quine—in favor of platonism rather than only for realism about mathematics. As noted above, though, given the arguments advanced by Putnam$_1$, which support the modal interpretation of mathematics and resist platonism, we can see that there are significant differences between Putnam's and Quine's indispensability arguments. But what exactly is Putnam's own argument then?

Perhaps the following tentative reconstruction will do:

(P'_1) All theories that quantify over objects that are indispensable to our best theories of the world ought to be interpreted realistically.

(P'_2) Mathematical theories quantify over objects that are indispensable to our best theories of the world.

Therefore, mathematical theories ought to be interpreted realistically.

In this formulation, we have an explicit argument for realism in mathematics, which does not presuppose or support platonism (an additional argument leading from realism to platonism is still required), and which highlights the significance played by mathematical objects in the sciences.

Someone may worry, however, that a realist interpretation of mathematics entails platonism. If a mathematical theory states that there are infinitely many sets and if that theory is interpreted realistically, doesn't it follow that sets exist, and hence that platonism is true? No, it doesn't. After all, as Putnam$_1$ would remind us, to interpret such mathematical theory realistically only requires that we take it as having a truth-value (bivalence is preserved), and given the modal interpretation of the resulting theory, verbal agreement with the platonist is reached without the commitment to the existence of mathematical objects. Only the possibility of the relevant structures is ever asserted.

It is interesting to note that Hermann Weyl, who initially developed a form of constructivism (in fact, predicativism) in his early work on the continuum (Weyl 1918[1994]), eventually revised his commitment to this view due to the amount of mathematics that he perceived as needed for the applications to physics (Weyl 1950). Putnam's indispensability argument, unlikely perhaps Quine's, would have resonated with Weyl.

There are, however, problems for the indispensability view. Is mathematics really indispensable to scientific theories? The most influential (and resisted!) response was given by Hartry Field (1980 and 1989). At least in the case of Newtonian gravitational theory, it is possible to dispense with the relevant mathematics.

But this requires a lot of hard work:

(i) Field argues that mathematical theories need not be true to be good, as long as they are conservative (that is, consistent with every consistent claim about the physical world). One needs to show that mathematics is conservative in this sense, which is something Field attempts to do (Field 1980, p.16–19; see also Field 1992).

Suppose a mathematical theory is conservative. It then follows that given a body *B* of nominalistic claims (claims about the world that do not refer to mathematical objects), if a nominalistic claim *N* follows from *B* together with some mathematics, *N* follows from *B* alone (that is, without any use of mathematics). The derivations involving mathematical objects, however, are typically shorter than those that do not. So, an important role of mathematics—the shortening of derivations—is highlighted.

But in order to have nominalistic premises to begin with, a second step is required in Field's program:

(ii) One needs to rewrite every physical theory without any quantification over mathematical objects, and show that the content of the original theory is properly preserved in its nominalistic counterpart. Field employs comparative predicates to this effect (Field 1980).

In fact, Field showed that this could be done for a particular physical theory: Newtonian gravitational theory (Field 1980, p.47–91).

The issue then arises as to whether additional theories, importantly different from Newtonian physics, such as non-relativist quantum mechanics, are amenable to the same nominalization strategy. It is unclear that they are, though (see Malament (1982)). While Field quantified over space-time regions in his nominalization of Newtonian gravitational theory, in the case of quantum mechanics, it is far less clear what the acceptable ontological basis for the reconstruction would be. Mark Balaguer (1998, Chapter 6) tried to implement a reconstruction of quantum mechanics along Field's lines, quantifying over potentialities. But there are serious worries as to whether such potentialities are nominalistically acceptable, and the proposal seems unable to get off the ground (see Bueno 2003).[10]

Is there an alternative? According to Jody Azzouni, we could grant that mathematics is indeed indispensable (thus granting the second premise of the indispensability argument, (P_2)), but resist the (Quinean) conclusion that this commits us to the existence of mathematical objects (Azzouni 2004). As opposed to Quine, Azzouni notes, we should distinguish two kinds of commitment (Azzouni 2004, p.49–122):

(i) *Quantifier commitment*: the commitment involved in the quantification over something. It is important that the quantifiers are interpreted as ontologically neutral, and thus no commitment to the existence of the objects that are quantified over is ever involved.

(ii) *Ontological commitment*: the commitment involved in taking something as existing. In order to express such commitment, an existence predicate is needed. So, to express the existence of electrons, 'Electrons exist', we would have: $\exists x(Lx \wedge Ex)$, where 'L' stands for the predicate 'is an electron' and 'E' stands for the existence predicate.

But what condition is the existence predicate supposed to satisfy? According to Azzouni, ontological independence provides this condition. Those objects that are ontologically independent of our linguistic practices and psychological processes (such as mountains and electrons) exist. In contrast, those objects that are just made up by us (such as fictional and mathematical entities) don't. Moreover, natural languages, on Azzouni's view, do not have an idiom that systematically expresses ontological commitment (in some cases the idiom does, in others it doesn't). Despite that, as opposed to previous forms of nominalism (such as Field 1980 and Hellman 1989), mathematical language, and in fact pretty much any discourse, can be taken literally.

This view, which can be called *deflationary nominalism* (see Bueno 2013), has a number of virtues. It allows one to preserve the objectivity of mathematics, given that after the introduction of certain mathematical axioms, it is not up to us what follows from such axioms. And given that the quantifiers are interpreted in an ontologically neutral way, the existence of mathematical objects is not assumed. Thus platonism is avoided, without the introduction of modal operators. In this way, deflationary nominalism *prima facie* seems able to realize much of the vision that Putnam had for the philosophy of mathematics, given that it seems to preserve realism and the objectivity of mathematics without platonism.

There are, however, difficulties for this proposal. I will mention two.[11] (a) As a nominalization strategy, deflationary nominalism makes nominalism just too easy to get. We now seem able to nominalize virtually everything we may want to. It is enough to quantify over these objects and deny that the existence predicate applies to them. Formulated in terms of ontological independence, the content of the existence predicate is somewhat malleable, in the sense that whether the predicate applies or not to a class of objects depends on the details of the philosophical views under consideration. In the particular case of mathematical ontology, it is controversial whether mathematical objects are taken to be ontologically independent from us or not. Platonists insist that such objects are indeed independent; thus, according to the deflationary nominalist's own criterion, mathematical objects exist. Nominalists, of course, deny that this is the case. Hence, there is significant controversy regarding whether certain contentious objects are ontologically independent of us or not. As a result, the strength of the ontological independence criterion is unclear, thus leaving room for selective choice of what class of objects is nominalized.

(b) Moreover, despite what has been advertised, deflationary nominalism seems to be revisionist and not able to take discourse literally. After all, some of natural language quantified locutions seem to be ontologically loaded. Consider, for example, the following cases:

(i) There are no witches.

This seems to make (the denial of) an existential claim, and thus, it seems to use the quantifier in an ontologically loaded way. Similarly, consider:

(ii) There are islands in South Florida.

This statement also seems to assert the existence of islands. In contrast, it's very odd to assert that:

(iii) "There are infinitely many prime numbers" is true, but prime numbers don't exist.

This claim, however, is perfectly coherent according to the deflationary nominalist, since quantification over objects (such as numbers) does not require their existence. As a result, since some quantified locutions seem to indicate ontological commitment (in some contexts, although not in all), in order to accommodate them, some level of revisionism is required by the deflationary nominalist view. At the very least, the statements need to be rewritten in order to incorporate explicitly the existence predicate.

The point here is to question the adequacy of the deflationary nominalist's claim to the effect that nothing in natural language marks ontological commitment. Some quantified expressions seem to (at least in suitable contexts, although not in all), as the examples above suggest. This doesn't entail, of course, that all natural language claims are ontologically loaded. Clearly, some aren't. Suppose that, while discussing British literature, I say:

(iv) There is a fictional detective who lived in Victorian London.

I am not here asserting the existence of a fictional character. I am simply quantifying over one. (Needless to say, the deflationary nominalist would have no difficulty to accommodate this example.) The worry here is that, underlying deflationary nominalism, we find in the end some revisionism—to accommodate existentially loaded claims—despite the insistence that discourse be taken literally.[12]

5. Conclusion

Given all of the difficulties examined above, perhaps Putnam was right in trying to combine the indispensability argument (Putnam_3)—thought of as an argument for the objectivity of mathematics rather than an argument for platonism—with modalism (Putnam$_1$). It's interesting that both modalism and the indispensability view ultimately aim to guarantee the objectivity of mathematics. But they achieve this result in very different ways: one depends on the success of modal translations, while the other depends on the success of science.

Since, as we saw, both modalism and the indispensability view have troubles, we end with a conclusion that Putnam, not surprisingly, has also reached (Putnam 1979[1994], p.499): "Nothing works"![13]

References

Asay, J. 2010. How to Express Ontological Commitment in the Vernacular. *Philosophia Mathematica* **18**: 293–310.

Azzouni, J. 2004. *Deflating Existential Consequence: A Case for Nominalism*. New York: Oxford University Press.

———. 2011. Nominalistic Content. In: Cellucci; Grosholtz; Ippoliti (eds.) (2011), p.33–51.

———. 2013. Hobnobbing with the Nonexistent. *Inquiry* **56**: 340–358.

Balaguer, M. 1998. *Platonism and Anti-Platonism in Mathematics*. New York: Oxford University Press.

Benacerraf, P.; Putnam, H. (eds.) 1983. *Philosophy of Mathematics: Selected Readings*. (Second edition.) Cambridge: Cambridge University Press.

Bueno, O. 2003. Is It Possible to Nominalize Quantum Mechanics? *Philosophy of Science* **70**: 1424–1436.

———. 2008. Nominalism and Mathematical Intuition. *Protosociology* **25**: 89–107.

———. 2009. Mathematical Fictionalism. In: Bueno; Linnebo (eds.) (2009), p.59–79.

———. 2013. Nominalism in the Philosophy of Mathematics. In: Edward N. Zalta (ed.), *Stanford Encyclopedia of Philosophy* (Fall 2013 edition).
URL = http://plato.stanford.edu/archives/fall2013/entries/nominalism-mathematics/

Bueno, O.; Linnebo, Ø. (eds.) 2009. *New Waves in Philosophy of Mathematics*. Hampshire: Palgrave MacMillan.

Cellucci, C.; Grosholtz, E.; Ippoliti, E. (eds.) 2011. *Logic and Knowledge*. Newcastle: Cambridge Scholars Publishing.

Chudnoff, E. 2013. *Intuition*. Oxford: Clarendon Press.

Colyvan, M. 2001. *The Indispensability of Mathematics*. New York: Oxford University Press.

Field, H. 1980. *Science without Numbers: A Defense of Nominalism.* Princeton, N.J.: Princeton University Press.

———. 1989. *Realism, Mathematics and Modality.* Oxford: Basil Blackwell.

———. 1992. A Nominalistic Proof of the Conservativeness of Set Theory. *Journal of Philosophical Logic* **21**: 111–123.

French, S.; Krause, D. 2006. *Identity in Physics: A Historical, Philosophical, and Formal Analysis*. Oxford: Clarendon Press.

Friedman, J. 2005. Modal Platonism: An Easy Way to Avoid Ontological Commitment to Abstract Entities. *Journal of Philosophical Logic* **34**: 227–273.

Gödel, K. 1964. What is Cantor's Continuum Problem? In: Benacerraf; Putnam (eds.) (1983), p.470–485.

Hellman, G. 1989. *Mathematics without Numbers: Towards a Modal-Structural Interpretation.* Oxford: Clarendon Press.

———. 1996. Structuralism without Structures. *Philosophia Mathematica* **4**: 100–123.

Liggins, D. 2008. Quine, Putnam, and the 'Quine-Putnam' Indispensability Argument. *Erkenntnis* **68**: 113–127.

Lewis, D. 1986. *On the Plurality of Worlds*. Oxford: Blackwell.

Malament, D. 1982. Review of Field, *Science without Numbers*. *Journal of Philosophy* **79**: 523–534.

Moore, G. 1982. *Zermelo's Axiom of Choice: Its Origins, Development and Influence*. New York: Springer-Verlag.

Parsons, C. 2008. *Mathematical Thought and Its Objects*. Cambridge: Cambridge U. P.

Pólya, G. 1954. *Mathematics and Plausible Reasoning.* (2 volumes.) Princeton: Princeton U. P.

———. 1962. *Mathematical Discovery.* (2 volumes.) New York: Wiley.

Putnam, H. 1967[1979]. Mathematics without Foundations. *Journal of Philosophy* **64**. Reprinted in Putnam (1979) p.43–59. Page references are to this volume.

———. 1971[1979]. *Philosophy of Logic*. New York: Harper and Row. Reprinted in Putnam (1979), p.323–357. Page references are to the latter.

———. 1975[1979]. What is Mathematical Truth? In: Putnam 1979, p.60–78.

———. 1979. *Mathematics, Matter and Method.* Philosophical Papers, volume 1. (Second edition.) Cambridge: Cambridge University Press.

———. 1979[1994]. "Philosophy of Mathematics: Why Nothing Works". *Proceedings of the Philosophy of Science Association* (Edited by P.D. Asquith and Henry Kyburg, Jr.) East Lansing: Philosophy of Science Association, 1979. (The title of the original publication was "Philosophy of Mathematics: A Report".) Reprinted in Putnam 1994, p.499–512. Page references are to the latter.

———. 1994. *Words and Life*. (Edited by James Conant.) Harvard: Harvard University Press.

———. 2012a. Indispensability Arguments in the Philosophy of Mathematics. In: Putnam (2012b), p.181–201.

———. 2012b. *Philosophy in an Age of Science: Physics, Mathematics, and Skepticism*. (Edited by Mario De Caro and David Macarthur.) Harvard: Harvard University Press.

Tieszen, R. 1989. *Mathematical Intuition: Phenomenology and Mathematical Knowledge*. Dordrecht: Kluwer Academic Press.

———. 2011. *After Gödel: Platonism and Rationalism in Mathematics and Logic*. Oxford: Clarendon Press.

Weyl, H. 1918[1994]. *The Continuum: A Critical Examination of the Foundation of Analysis*. (Translated by Stephen Pollard and Thomas Bole.) New York: Dover.

———. 1950. *Philosophy of Mathematics and Natural Science*. (Second edition.) Princeton: Princeton University Press.

Otávio Bueno
Department of Philosophy
University of Miami
Coral Gables, FL 33124, USA
otaviobueno@me.com

Resumo. Nesse trabalho, examino as concepções sutis que Putnam desenvolveu em filosofia da matemática, distinguindo três propostas: o modalismo (uma interpretação da matemática em termos de lógica modal), o realismo quase-empírico (que enfatiza o papel e o uso de métodos quase-empíricos na matemática), e uma concepção indispensabilista (que salienta a função indispensável da quantificação sobre objetos matemáticos e o apoio proporcionado por tal quantificação para uma interpretação realista da matemática). Argumento que, ao longo dessas mudanças, Putnam buscou preservar um realismo semântico acerca da matemática que evite o platonismo. Ao final, todavia, cada uma das concepções propostas enfrenta dificuldades significativas. Uma forma de ceticismo então surge.

Palavras-chave: Putnam; argumento da indispensabilidade; modalismo; filosofia da matemática; conjunto; Quine.

Notes

[1] For a detailed development of such a modal-structural interpretation of mathematics, see Hellman 1989 and 1996. In what follows, I will adopt, in outline, Hellman's formulation, which is supposed to be a proper articulation of Putnam's conception.
[2] I would like to thank Hartry Field for pressing this point.
[3] Additional challenges to the possibility of drawing the distinction between nominalistic and mathematical content are raised in Azzouni 2011.
[4] For additional discussion of mathematical intuition, see Tieszen 1989 and 2011 (and references therein), Parsons 2008, and Chudnoff 2013.
[5] A fascinating account of the introduction and development of this axiom is found in Moore 1982 (see the references therein too).
[6] I owe this point to Catherine Elgin.
[7] For an account of mathematical intuition that does not rely on platonism, see Bueno 2008.
[8] Catherine Elgin rightly pointed this out to me.
[9] This point is made very clearly in Liggins 2008, and, of course, in Putnam 2012a.
[10] For additional critical discussion of Field's program, see Bueno 2013.
[11] Additional critical discussion of deflationary nominalism can be found in Bueno 2013.
[12] The deflationary nominalist may acknowledge that some uses of 'there are' are ontologically committing, while others aren't (see Azzouni 2013, p.344–5). What is needed then is a principled account of when such usages are operative and when they aren't—while still allowing discourse to be taken literally. It is unclear, however, how this can be done given the need to introduce the existence predicate. It has been suggested that ontological commitment in the vernacular can be expressed via negative existential sentences (Asay 2010). But this is not a view the deflationary nominalist could—or would want to—endorse in general, given the points made in Azzouni 2013.
[13] An earlier version of this work was presented at the VIII Principia Symposium on the Philosophy of Hilary Putnam (in Florianópolis, Brazil, in August, 2013). Thanks go to the organizers of the meeting, in particular to Cezar Mortari, for providing the occasion for me to write this paper and for the extremely congenial and stimulating event they created. My thanks are also due to Jody Azzouni, Catherine Elgin, Hartry Field, Dagfinn Føllesdal, Guillermo Rosado Haddock, and Décio Krause for their feedback and extremely helpful discussions.