PRINCIPIA 27(3): 519–538 (2023) doi: 10.5007/1808-1711.2023.e92106
Published by NEL — Epistemology and Logic Research Group, Federal University of Santa Catarina (UFSC), Brazil.

ON VALIDITY PARADOXES AND (SOME OF) THEIR SOLUTIONS

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Abstract. Many semantic theories become trivial when extended with a naïve validity predicate due to the validity paradoxes. The non-classical semantic theories are the ones that allegedly preserve the *naïveté* of the validity predicate while being capable of avoiding the validity paradoxes. This blocking, on the other hand, usually comes at a high cost. In this paper, we argue that the pre-theoretical notion of validity that the naïve validity predicate intends to capture is unattainable.

Keywords: paradoxes • predicates • non-classical logic • logical validity • philosophy of logic

RECEIVED: 09/12/2022 REVISED: 04/10/2023 ACCEPTED: 06/10/2023

1. Introduction

Logic, in its contemporary understanding, is a theory of consequence relation. In this sense, logic investigates what arguments are, in fact, valid. The centrality of the notion of validity justifies the philosophical investigations of its general properties. From this general perspective, one intends to analyze a more intuitive/pre-theoretical notion of validity, which incorporates the deductive inferences of formal logic and deductive inferences present in scientific reasoning, analytic inferences, and so on. The most straightforward way to investigate its properties is to consider validity as a predicate and introduce it into a theory with enough expressive resources to talk about its own sentences. Such an approach is usually called *naïve approach* to validity. A naïve theory of validity is a theory that contains its own notion of validity (Barrio et al. 2016).

However, the validity paradoxes threaten the naïve approach to validity. Since an inconsistency makes any theory based on classical logic trivial, such an approach is usually considered incompatible with classical logic, as the naïve theory of truth is so (Tarski, 1956). Given the emergence of the inconsistency results about the naïve validity predicate, the non-classical solutions were proposed in order to block the problematic steps in the derivation of the paradoxes and save as much as possible the properties of the naïve validity predicate. Such non-classical solutions are obtained by dropping out some classical principles that usually make presence in the proof of these problematic results.

As we will see, this notion of validity expressed by the naïve predicate is problematic. In this paper, we will defend that the notion of logical validity, although relative to the logical system, is the best we can obtain, and we will argue against the existence of a notion of validity that is pre-theoretic and has an absolute character. This paper is structured as follows: Section 2 presents the validity paradoxes and their philosophical significance. Section 3 presents classical and non-classical solutions to the paradoxes of validity. We present Ketland's (2012) and Skyrms's (1978) solutions in the classical case. In the non-classical case, we present Pailos's (2020) and Barrio et al.'s (2016) solutions. Even if these solutions are not exhaustive, we argue in Section 4 that these solutions manifest general issues relevant to our discussion. In this section, we also argue for the logical validity predicate. In Section 5, we close the discussion with a few remarks.

2. Validity and its paradoxes

In what follows, we present some inconsistency results about the concept of validity. First, we present Montague's result about incorporating of modal predicates in the arithmetical language. As we will see, since the naïve validity predicate obeys some modal principles, Montague's theorem also holds for the naïve approach to validity. Second, we present a generalization of this result known as *v-Curry*. We also discuss the philosophical significance of such results.

2.1. Montague's theorem and validity predicate

From a broad perspective, the validity of a sentence φ can be understood as "giving compelling logical grounds to believe φ" (Halldén, 1963). This notion is general enough to encompass mathematical, logical, and analytic inferences. Let $Val(x)$ be the validity predicate. A theory T of validity aims to establish the most general principles governing the predicate Val. This predicate is expected to satisfy the following minimal principles:[1]

(Val-D) $Val(\ulcorner\varphi\urcorner) \to \varphi$;

(Val-In) Given a valid derivation of φ, infer $Val(\ulcorner\varphi\urcorner)$.

Where $\ulcorner\varphi\urcorner$ stands for the arithmetical name of φ. The principle (Val-D) is a *detachment* principle of Val. That is, if φ is valid, then it is the case that φ.[2] (Val-In) is an *introduction* principle of the predicate Val. Given a valid derivation of φ, we obtain that φ is valid in the sense of Val. In the analysis of predicates such as Val, PA is usually the underlying syntax theory because it allows one to talk about its sentences, language, and so on. So:

Definition 2.1. Let $\mathscr{L}_{\mathsf{PA}}$ be the language of PA. The theory of validity T is obtained by extending the $\mathscr{L}_{\mathsf{PA}}$ with the predicate Val that satisfies the principles (Val-D) and (Val-In).

The high expressive power of the language of T comes with a cost. It is a well-known fact that PA proves Diagonalization Lemma. So, by Diagonalization Lemma, one obtains the following sentence:

$$\varphi \leftrightarrow \neg Val(\ulcorner \varphi \urcorner) \tag{1}$$

Then, as the following version of Montague's theorem shows, T is inconsistent.

Theorem 2.2. *(Montague 1963) Suppose that T is a theory such that:*

1. *T is an extension of arithmetic* PA*;*[3]
2. $\vdash_T Val(\ulcorner \varphi \urcorner) \rightarrow \varphi$*;*
3. *If* $\vdash_T \varphi$ *then* $\vdash_T Val(\ulcorner \varphi \urcorner)$*.*

Then, T is inconsistent.

Proof. (Sketch) Given the sentence $\varphi \leftrightarrow \neg Val(\ulcorner \varphi \urcorner)$, plus schema (2) and rule (3), we obtain $Val(\ulcorner \varphi \urcorner) \wedge \neg Val(\ulcorner \varphi \urcorner)$ by employing some propositional reasoning. Then, T is inconsistent. Q.E.D.

An informal version of Theorem 2.2 can also be found in Myhill (1960), where Val stands for *absolute provability* of arithmetical sentences. Montague's Theorem also has a philosophical importance in the early discussion about the legitimacy of modal logics. As it is known, Quine was one of the main critics of using modal logics. Quine (1966) outlined three ways we can be involved with modalities. The first concerns the use of modalities as predicates (syntactical approach); the second concerns the use of modalities as sentence operators; and the third concerns the use of modalities as operators in first-order modal logic. Among these three ways, Quine argues that the first is the least pernicious since some predicates of sentences that are, according to him, legitimate, such as the *theoremhood* predicate. Since the conditions 2 and 3 of Theorem 2.2 are characteristic principles of many modal systems, Montague argues that if modalities are to be conceived as predicates, then "all of modal logic, even the weak system S1, must be sacrificed" (Montague 1963, p.154).

2.2. Curry's paradox for validity

Theorem 2.2 is not the only problem that the naïve approach to validity faces. As Beall and Murzi (2013) show, Val can be generalized to arguments as follows:

$Val(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)$ means that there is a valid inference of ψ from φ.

Let Γ and Δ be sets of formulas. Such a predicate, generally presented through sequent calculus, is expected to satisfy the following properties:

$$\dfrac{\varphi\Rightarrow\psi}{\Rightarrow Val(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)}\text{(Val-In)} \qquad \dfrac{\Gamma\Rightarrow\varphi \qquad \Delta\Rightarrow Val(\ulcorner\varphi\urcorner,\ulcorner\psi\urcorner)}{\Gamma,\Delta\Rightarrow\psi}\text{(Val-D)}$$

The unary predicate Val can be defined as $Val(\ulcorner\varphi\urcorner) := Val(\ulcorner\top\urcorner,\ulcorner\varphi\urcorner)$. Beall and Murzi generalize Montague's result in such a way that it will depend much less on the rules of the propositional connectives, such as $\neg$, but on some structural rules and on the detachment of $\rightarrow$, which are given as follows:

$$\dfrac{}{\varphi\Rightarrow\varphi}(Ref.) \qquad \dfrac{\Gamma,\varphi,\varphi\Rightarrow\psi}{\Gamma,\varphi\Rightarrow\psi}(L.Contr.)$$

$$\dfrac{\Gamma\Rightarrow\varphi \qquad \Delta,\varphi\Rightarrow\psi}{\Gamma,\Delta\Rightarrow\psi}(Cut) \qquad \dfrac{\Gamma\Rightarrow\varphi \qquad \Delta\Rightarrow\varphi\rightarrow\psi}{\Gamma,\Delta\Rightarrow\psi}(\rightarrow\text{-E})$$

Now, let T be an arithmetical theory closed by the sequent rules (Val-In), (Val-D), ($\rightarrow$-E) and by the above structural rules. Again, by Diagonalization Lemma, one obtains:

(2) $$\pi\leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)$$

Then the following result follows:

Theorem 2.3. *(Beall and Murzi 2013) T is trivial.*

Proof. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\pi\Rightarrow\pi}\text{(Ref.)} \qquad \dfrac{}{\Rightarrow\pi\leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)}\text{(2)}}{\pi\Rightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)}(\rightarrow\text{-E}) \qquad \dfrac{}{\pi\Rightarrow\pi}\text{(Ref.)}}{\pi,\pi\Rightarrow\bot}\text{(Val-D)}}{\pi\Rightarrow\bot}\text{(L.Contr.)}}{\Rightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)}\text{(Val-In)}$$

Call this derivation Π. Then:

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow\pi\leftrightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)}\text{(2)} \qquad \begin{array}{c}\Pi\\ \Rightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)\end{array}}{\Rightarrow\pi}(\rightarrow\text{-E}) \qquad \begin{array}{c}\Pi\\ \Rightarrow Val(\ulcorner\pi\urcorner,\ulcorner\bot\urcorner)\end{array}}{\Rightarrow\bot}\text{(Val-D)}$$

This concludes the proof. Q.E.D.

Theorem 2.3 is known in the literature as *validity-Curry* (*v-Curry*, for short). v-Curry has an interesting philosophical significance because it poses a problem to the non-classical solutions to logical paradoxes, especially to the paracomplete and paraconsistent approaches, which are usually adopted in order to restrict the behaviour of the negation operator. Since negation plays no role in the derivation of v-Curry, the legitimacy of such non-classical solutions becomes questionable. There are paraconsistent and paracomplete solutions to v-Curry, but they are not just about restricting the behavior of the negation connective.

3. Solutions to validity paradoxes

3.1. Classical solutions

Here we analyze two classical solutions to the validity paradoxes: one given by Skyrms (1978) and one by Ketland (2012). Both proposals originally presented their results in terms of the unary validity predicate Val. We will keep their original notations, but we note that their results can be easily converted to a sequent presentation. Moreover, the binary validity predicate can be defined in their theories in terms of the unary predicate as: $Val(\overline{\varphi}, \overline{\psi}) := Val(\overline{\varphi \to \psi})$ (Skyrms's proposal) and $Val(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) := Val(\ulcorner \varphi \to \psi \urcorner)$ (Ketland's proposal).[4]

3.1.1. Skyrms's solution

In Skyrms (1978), Skyrms proposes to establish a connection between the concepts of necessity and validity. His work is twofold. First, he proposes a hierarchical solution to Montague's theorem.[5] Second, he shows that his hierarchical construction satisfies some familiar modal principles. Informally, his construction goes as follows. Let T be a theory which extends at least classical propositional logic (CPL), and $\mathscr{L}_T$ the language of T. The hierarchical language relative to T takes $\mathscr{L}_T$ as the base language, which we denote by $\mathscr{L}_T^0$ for convenience. Informally, given $\mathscr{L}_T^0$, we construct a hierarchy of increasingly stronger languages $\mathscr{L}_T^1, \mathscr{L}_T^2, \ldots, \mathscr{L}_T^i, \ldots$, where each $\mathscr{L}_T^i$, for $i > 0$, contains the formulas φ of the languages $\mathscr{L}_T^k$, $0 \leq k < i$, as well as sentence names $\overline{\varphi}$ for each φ and the predicate Val. Formally:

Definition 3.1. (Skyrms 1978) Let $\mathscr{L}_T^0$ be a language of the theory T which contains at least the language of CPL. From $\mathscr{L}_T^0$, we construct the languages $\mathscr{L}_T^M$ and $\mathscr{L}_T^\omega$ as follows:

1. The language $\mathscr{L}_T^M$:

1. If φ is a sentence of $\mathscr{L}_T^0$, then φ is a sentence of $\mathscr{L}_T^M$.
2. If φ and ψ are sentences of $\mathscr{L}_T^0$, then $\neg\varphi$, $\varphi \to \psi$ and $\Box\varphi$ are sentences of $\mathscr{L}_T^M$.

3. The languages $\mathscr{L}_T^n$'s:

1. If φ is a sentence of $\mathscr{L}_T^n$ and $\overline{\varphi}$ is a sentence name, then φ and $Val(\overline{\varphi})$ are sentences of $\mathscr{L}_T^{n+1}$.
2. If φ and ψ are sentences of $\mathscr{L}_T^{n+1}$, then $\neg\varphi$ and $\varphi \to \psi$ are sentences of $\mathscr{L}_T^{n+1}$.

The set of sentences of $\mathscr{L}_T^\omega$ is the union of the set of sentences of $\mathscr{L}_T^n$'s, for $n \in \omega$

From the models for the language $\mathscr{L}_T^0$ we define the models for $\mathscr{L}_T^\omega$ as follows:

Definition 3.2. (Skyrms 1978) Let v^0 be a model for the language $\mathscr{L}_T^0$. A model v^0 induces a chain of models v^n for the languages $\mathscr{L}_T^n$'s as follows:

(1) The model v^0 of $\mathscr{L}_T^0$ is the model v for the language $\mathscr{L}_T$.

(2) The model v^{n+1} of $\mathscr{L}_T^{n+1}$ is induced by a model v^0 of $\mathscr{L}_T^0$ is the smallest extension of v^n of $\mathscr{L}_T^n$ such that:

(2.1) $v^{n+1}(Val(\overline{\varphi})) = 1$ if $v'^n(\varphi) = 1$ for all models v'^n of $\mathscr{L}_T^n$; otherwise $v^{n+1}(Val(\overline{\varphi})) = 0$;

(2.2) The interpretation of $\neg$ and $\to$ are given by the truth-tables of T.

The model v^ω of the language $\mathscr{L}_T^\omega$ induced by the model v^n of $\mathscr{L}_T^n$ is the union of the models v^n of $\mathscr{L}_T^n$.

By defining the translation $t : For(\mathscr{L}_T^M) \to For(\mathscr{L}_T^w)$ such that $t(\varphi) = \varphi$ for $\varphi \in For(\mathscr{L}_T)$ and $t(\Box\varphi) = Val(\overline{t(\varphi)})$, Skyrms proves that the models of the Definition 3.2 validate the principles of propositional S5.

Now, let $T = \mathsf{PA}$ and call PA^ω the theory obtained by extending the language of PA in the lines of Definition 3.1. So, how does PA^ω block the above paradoxical results? The sentence $\varphi \leftrightarrow \neg Val(\ulcorner\varphi\urcorner)$ is blocked PA^ω due to a severe syntactical restriction on the behavior of the predicate Val. That is, since only sentence names occur in the predicate Val, it is not obvious that Diagonalization Lemma applies to this predicate. The same reasoning applies to the Curry sentence. Thus, Skyrms's validity theory blocks the paradoxes of validity by blocking any possibility of diagonalizing on the predicate Val. So, if PA is consistent, then so is PA^ω.

Even if Skyrms's solution blocks both paradoxes, it may not be a satisfactory solution from a philosophical perspective. As Stern (2015) observes, the restriction imposed on the predicate Val in Definition 3.1 is so severe that one can claim that

Val is a modal operator in disguise since only sentence names can be arguments of Val. As we argued before, the Diagonalization Lemma does not apply to this predicate since only sentence names occur in Val. These restrictions imposed by Skyrms's solution imply that the resulting language is philosophically unappealing because it prohibits the possibility of diagonalizing on Val (Egré 2005).

It is possible to modify Skyrms's proposal in order to make it stronger. Indeed, Hazen (1984) shows that it is possible to extend consistently Skyrms's metalanguage with a function Q such that, for every φ, $Q(\varphi) = \overline{\varphi}$. By doing so, we have the following version of Diagonalization Lemma:

Lemma 3.3. *(Otte 1982) Let T be a theory extending* PA *with the predicate Val in the lines of Definition 3.1. For any $\psi(x)$ of T with a free variable x there is a sentence φ such that $\vdash_T \varphi \leftrightarrow \psi(x)$.*

By Lemma 3.3, we can obtain the sentence $\varphi \leftrightarrow \neg Val(\overline{\varphi})$, and then deriving Theorem 2.2. To avoid this problem, according to Hazen (1984), the predicate Val must to be contextually definable, in the sense that each level $\mathscr{L}_T^n$ would have its particular validity predicate Val_n which talks about validity of the sentences of $\mathscr{L}_T^k$, for $k < n$. Of course, both Definitions 3.1 and 3.2 will have to be modified as follows. In the case of Definition 3.1, we have:

(a') If φ is a sentence of $\mathscr{L}_T^n$ and $\overline{\varphi}$ is a sentence name, then φ and $Val_{n+1}(\overline{\varphi})$ are sentences of $\mathscr{L}_T^{n+1}$.

In the case of Definition 3.2, we have:

(2.1') $v^{n+1}(Val_{n+1}(\overline{\varphi})) = 1$ if $v'^n(\varphi) = 1$ for all models v'^n of $\mathscr{L}_T^n$;
otherwise $v^{n+1}(Val_{n+1}(\overline{\varphi})) = 0$;

In this scenario, even if we obtain by Diagonalization $\varphi \leftrightarrow \neg Val_n(\overline{\varphi})$ in the language $\mathscr{L}_T^n$, we do not obtain $\vdash_T Val_n(\overline{\varphi}) \wedge \neg Val_n(\overline{\varphi})$ because from $\vdash_T \varphi$ we can only obtain $\vdash_T Val_{n+1}(\overline{\varphi})$. Obviously, $\vdash_T \neg Val_n(\overline{\varphi}) \wedge Val_{n+1}(\overline{\varphi})$ is not a contradiction. This hierarchical construction is close to Anderson's proposal (Anderson 1983) for the knowability predicate.

3.1.2. Ketland's solution

Skyrms's solution to Montague's Theorem proceeds by a severe syntactical restriction on the validity predicate in such a way that we cannot obtain the sentence $\varphi \leftrightarrow \neg Val(\overline{\varphi})$. As Ketland (2012) shows, it is possible to use the arithmetical names occurring in Val, so long as some restrictions in the behavior of Val are to be made. In his characterization of such a predicate, Ketland considers the schemas (Val-K) and (Val-D) as the axioms for the predicate Val with the following introduction rule of this predicate:

(Val-K) $Val(\ulcorner \varphi \to \psi \urcorner) \to (Val(\ulcorner \varphi \urcorner) \to Val(\ulcorner \psi \urcorner))$;

(Val-D) $Val(\ulcorner \varphi \urcorner) \to \varphi$;

(Val-In') Given a *logical derivation* of φ, infer $Val(\ulcorner \varphi \urcorner)$.

A *logical derivation* is a derivation that uses only the logical axioms and rules of the formal system. In the present case, he adopts the deductive system of First-Order Logic (FOL). Since FOL is complete, every (logically) provable formula is logically valid. In this sense, the predicate Val can be introduced only in the conclusions φ of a derivation. So, given a logical derivation $\langle \varphi_1, \ldots, \varphi_n, \psi \rangle$, we can conclude that $\langle \varphi_1, \ldots, \varphi_n, \psi, Val(\ulcorner \psi \urcorner) \rangle$ since ψ is derivable (i.e. valid). In his mentioned above paper, the formal system that formalizes Val is called *V-logic* (hereafter VL), defined as follows:

Definition 3.4. VL is obtained by extending PA with predicate Val satisfying (Val-K), (Val-D) and (Val-In').

Given that FOL-provability can be encoded within PA, VL is a conservative extension of PA. So, if PA is consistent, then so is VL. According to Ketland, logical validity is not susceptible to the inconsistencies proved by Theorems 2.2 and 2.3 because these proofs are not logical proofs but proofs that use arithmetical resources of PA, such as the Diagonalization Lemma. So, for example, we cannot apply the rule (Val-In') in the last step of the derivation Π of Theorem 2.3. Consequently, assertions concerning logical validity are not logically valid in the sense of being provable in the basic logical system. As Cook (2014) observes, VL does not prove substitutivity of equivalents. To see this, consider the formula $Val(\ulcorner \varphi \urcorner) \leftrightarrow Val(\ulcorner \neg\neg\varphi \urcorner)$. Since this formula is not a FOL-theorem, we cannot apply the rule (Val-In') to this formula. Then, validity assertions are not themselves valid.

As we can see, both Skyrms's and Ketland's solutions adopt strategies to block Diagonalization Lemma in some way: while Ketland restricts the introduction of Val to logical proofs, Skyrms restricts the syntax of Val to primitive sentence names. The latter solution faces the difficulty of justifying the restricted behavior of predicate Val, whereas the former can be criticized for dealing with a restricted kind of validity since logical validity does not exhaust naïve validity. Now, we will present some nonclassical solutions.

3.2. Non-classical solutions

Non-classical logics figure out interesting tools for dealing with paradoxes. As Murzi and Carrara (2015) observe, such logical revision should go through *substructural logics*, which abandon some structural rules, since semantic theories based on non-classical structural logics are usually subjected to v-Curry.[6] In what follows, we will analyze two non-classical solutions: a structural solution given by Pailos (2020) and a substructural solution given by Barrio et al. (2016). Even if we concentrate on two non-classical solutions, we will argue that the other solutions face similar problems.

3.2.1. A structural approach to validity

As argued before, v-Curry makes no use of the negation connective, posing a problem to paraconsistent and paracompleteness upholders. However, it is still possible to adopt these logics to block v-Curry. For example, Field (2017) proposes a validity theory based on a paracomplete logic that drops out the rule (Val-D) by arguing that it contradicts Gödel's Second Incompleteness Theorem. So, his approach blocks v-Curry, given that its derivation makes essential use of the rule (Val-D). Here, we will analyze a proposal that maintains both rules of validity predicate. In (Pailos 2020), Pailos proposes a paraconsistent solution to v-Curry based on the first-order *Logic of Paradox* (LP, Asenjo 1966, Priest 1979) whose interpretation of the connectives will be given below. The theory LP^{Val} is obtained by introducing, for every sentence φ, its *designated name* $\langle\varphi\rangle$ and the binary predicate Val. Now, consider the following definition.

Definition 3.5. LP^{Val} is a first-order theory of validity whose propositional operations and predicate Val respect the following tables:

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\vee$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$Val(\langle\varphi\rangle, \langle\psi\rangle)$	1	$\frac{1}{2}$	0
1	1	1	0
$\frac{1}{2}$	1	1	0
0	1	1	1

where 1 and $\frac{1}{2}$ are designated values. The notions of *tautology*, *logical validity*, and *logical consequence* are defined as usual for LP (Priest 2008).

The validity predicate is interpreted as the implication connective of Sette's three-valued paraconsistent logic P^1 (Sette 1973). According to the table of Val, validity predicate only receives classical values because "there is no inference that is neither (both) valid nor (and) invalid" (Paillos 2020, p.778). Pailos's proposal is closely tied to Goodship's project (1996), according to which detachment-free conditionals are one of the keys to obtaining non-trivial semantic theories.[7]

With the aid of the designated names $\langle\varphi\rangle$, Pailos introduces a *weak self-referential principle* for LP^{Val}, which is stated as follows:

Definition 3.6. Let T be a theory that has a name forming device $\langle\cdot\rangle$. If for every formula $\psi(x)$, with x as the only free variable in $\psi(x)$, there is a (closed) formula φ such that the formula $\varphi \leftrightarrow \psi(\langle\varphi\rangle)$ is true in T, then we say that T adopts a *weak self-referential procedure* (WSRP).

Since WSRP is introduced from "outside the theory" it is necessary to find a way to validate self-referential statements in LP^{Val}. Let $x^* \leftrightarrow \psi_{x^*}$ be a sentence scheme, where $*$ is a metalinguistic mark, x^* is a distinguished propositional letter, and ψ_{x^*} is any sentence with at least one instance of $Val(\langle x^P\rangle, \langle x^C\rangle)$ such that x^* is a subformula either of x^P or x^C or of both. Moreover, he considers a function Q such that for every formula φ, $Q(\varphi) = \langle\varphi\rangle$. Last, let Z be an infinite set of such biconditionals which receive only designated values. Then, every member of Z is an instance of WSRP. To see that such set exists, just assign $\frac{1}{2}$ to every propositional letter.

Now, consider the Curry sentence $p \leftrightarrow Val(\langle p\rangle, \langle\bot\rangle)$. To see that such sentence does not trivialize LP^{Val}, consider a model v which assigns $\frac{1}{2}$ to p. By the definition of $\leftrightarrow$, we obtain that $v(p \leftrightarrow Val(\langle p\rangle, \langle\bot\rangle)) = \frac{1}{2}$. LP^{Val} is also immune to Theorem 2.2 due to the behavior of $\neg$. Moreover, Pailos shows that LP^{Val} is non-trivial.

It is important to note the role of WSRP in LP^{Val}. As Pailos emphasizes, the presence of a stronger self-reference principle makes LP^{Val} trivial. Consider the following self-reference principle:

Definition 3.7. Let T be a theory that has a name forming device $\langle\cdot\rangle$. If for every formula $\varphi(x)$, with x as the only free variable, there is a term t such that t is identical to the name of $\varphi(t)$, then we say that T adopts a *strong self-referential procedure* (SSRP).

In his paper, Pailos shows that SSRP makes LP^{Val} trivial. This shows that even a rather weak semantic theory is threatened by v-Curry.[8] To block v-Curry in a validity theory based on a structural logic, then it is necessary to (i) adopt a hierarchical approach to validity (Whittle 2004) or (ii) to drop, or even restrict, (Val-In) or (Val-D) (Field 2017).

3.2.2. A substructural approach to validity

The substructural solutions to validity paradoxes have become wide-spread in the literature of philosophical logic. In a certain way, such solutions usually take some structural rules of the sequents as the source of the paradoxes of validity. That is, such solutions block some principles about inferences to deal with the problematic steps in the derivation of the paradoxes. For example, there are non-reflexive theories of validity (*e.g.*, Meadows 2014, Murzi and Rossi 2017), non-contractive validity theories (*e.g.*, Zardini 2013, Weber 2014). Here, we will focus on the Cut-free validity theories. Cut-free approaches to logical paradoxes have been recently explored by Cobreros et al. 2012, Ripley 2012; 2013 and Barrio et al. 2016.

A common objection to non-classical logics concerns their informal interpretation. In the context of non-classically based theories of truth, Terzian (2015) argues that logical revision is not always accompanied by an intuitive explanation of the notion it intends to formalize. In this sense, the naturalness of these solutions may be called into question. However, Dutilh Novaes and French (2018) argue that substructural solutions to logical paradoxes can have a nice dialogical interpretation, except for the Cut-free approaches. However, as we will see, Cut-free approaches make sense in terms of a bilateralist interpretation of the consequence relation. The naïve approach to validity based on the Cut-free approach we will analyze here is proposed by Barrio et al. (2016), and it is based on *Strict Tolerant Logic* (ST, for short). Consider the following definition:

Definition 3.8. Let Γ and Δ be multisets. The theory ST^{Val} is presented by the following sequent rules:

$$\frac{}{\varphi \Rightarrow \varphi}(Ref)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}(LW) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}(RW)$$

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}(LContr) \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}(RContr)$$

$$\frac{\Gamma, \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}(L\neg) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}(R\neg)$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}(L\wedge) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}(R\wedge)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}(L\vee) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}(R\vee)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}(L \to) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}(R \to)$$

$$\frac{}{\Gamma, Val(\langle \bigwedge \Gamma \rangle, \langle \bigvee \Delta \rangle) \Rightarrow \Delta}(\text{Val-D'}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Rightarrow Val(\langle \bigwedge \Gamma \rangle, \langle \bigvee \Delta \rangle)}(\text{Val-In})$$

where $\bigwedge \Gamma$ (resp., $\bigvee \Delta$) denotes the conjunction (resp., disjunction) of all members of Γ (resp., Δ). The notion of proof is defined as usual in sequent calculus.

Ripley (2013) argues that ST is compatible with the bilateralist interpretation of logical consequence. According to such an interpretation, sequents like $\varphi \Rightarrow \psi$ are interpreted in terms of assertability and deniability.[9] That is, $\varphi \Rightarrow \psi$ means that asserting φ and denying ψ is "out of bounds." As a consequence, the validity theory based on this logic is to be understood in terms of this bilateralist interpretation.

As one may observe, the rule (Val-D') is formulated differently than our earlier presentation. Such a difference is not innocuous. According to Barrio et al., the rule (Val-D) presupposes some form of Cut. Then, in a Cut-free approach to validity, adopting the rule (Val-D') is more adequate.

The logic ST has some interesting properties. For example, it can be proved that ST has the same set of tautologies as CPL and the same set of valid inferences (Barrio et al. 2015). Their difference lies in the *metainferences* they validate, which are inferences between inferences. ST^{Val} is a Cut-free theory, and since the derivation of v-Curry makes essential use of the rule of Cut, then ST^{Val} blocks it because such a rule is not available in the formal system.

As is expected of any theory of validity, ST^{Val} is expected to express the validity of its inferences. For example, ST^{Val} should prove the following facts:

$$\Rightarrow Val(\ulcorner \varphi \land \psi \urcorner, \ulcorner \varphi \urcorner) \qquad \text{(Conj. elim. A)}$$
$$\Rightarrow Val(\ulcorner \varphi \land \psi \urcorner, \ulcorner \psi \urcorner) \qquad \text{(Conj. elim. B)}$$
$$\Rightarrow (Val(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \land Val(\ulcorner \varphi \urcorner, \ulcorner \gamma \urcorner)) \to Val(\ulcorner \varphi \urcorner, \ulcorner \psi \land \gamma \urcorner) \qquad \text{(Val-Conj. intr.)}$$

The possibility to prove the validity of rules such as (Conj. elim. A) and (Conj. elim. B) and of the metarule (Val-Conj. intr.) shows that ST^{Val} is capable to *internalize* these rules and the *metarule* (*i.e.*, inference rules between inferences). More precisely, internalization is defined as follows:

Definition 3.9. (Barrio et al. 2016) We say that a theory T *internalize* a metarule $\mathfrak{r}$ of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}(\mathfrak{r})$$

if T proves every instance of

$$\overline{\Rightarrow (Val(\bigwedge\Gamma_1, \bigvee\Delta_1) \wedge \ldots \wedge Val(\bigwedge\Gamma_n, \bigvee\Delta_n)) \rightarrow Val(\bigwedge\Gamma, \bigvee\Delta)}$$

However, as Barrio et al. prove, ST^{Val} fails in internalizing some of its metarules. Their counterexample is the formula $Val(\ulcorner\varphi\urcorner, \ulcorner\gamma \vee \psi\urcorner) \rightarrow Val(\ulcorner\varphi \wedge \neg\gamma\urcorner, \ulcorner\psi\urcorner)$. As they show, ST^{Val} does not prove such a formula. In their paper, they present some attempts to strengthen the predicate of validity by adding stronger introduction and detachment rules for Val. But, as they show, such a task is far from being simple because there is the possibility of regaining some form of the rule of Cut, which would constitute a problem for the upholder of non-transitive approaches to validity. So, we have the following problem: either we accept that non-transitive approaches to validity fail in internalizing their validities, or we strengthen the validity theory based on ST with stronger rules for Val and then internalize Cut. As Rosenblatt (2017) shows, many substructural approaches to naïve validity also fail in internalizing all their metarules. The importance of internalization is, first of all, philosophical: it is reasonable to expect that the rules and metarules of the logical system should be naïvely valid. Second, from a local perspective about validity, Val is expected to interact with the other connectives in the language of the theory T. Take the provability predicate $Prov_{\mathsf{PA}}$ as an example. Since classical conditional $\rightarrow$ validates modus ponens, it is reasonable to expect that $Prov_{\mathsf{PA}}(\ulcorner\varphi \rightarrow \psi\urcorner) \rightarrow (Prov_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \rightarrow Prov_{\mathsf{PA}}(\ulcorner\psi\urcorner))$ should be provable.

In (2019), Hlobil proposes the requirement of *faithfulness*, which can be stated as follows:

Definition 3.10. A validity predicate, Val, is *faithful* just in case $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), \ldots, Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \Rightarrow Val(\langle\Pi\rangle, \langle\Theta\rangle)$ is provable iff $\Pi \Rightarrow \Theta$ follows from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$.

In his paper, Hlobil shows that the requirement of faithfulness is enough to solve the dilemma posed by Barrio et al. (2016). Faithfulness establishes that a Cut-free validity theory based on ST internalizes all its validities to the cost of abandoning the rule (Val-D). He proves that faithfulness plus contraction and Curry sentences are incompatible with such rule of detachment.

Even if Hlobil's proposal shows that non-transitive approaches to naïve validity can face Barrio et al.'s dilemma, it does not seem to be a good idea to dispense the rule (Val-D). One of the most common arguments to dispense (Val-D) is that $\top, Val(\langle\top\rangle, \langle\varphi\rangle) \Rightarrow \varphi$ contradicts Löb's theorem (Löb 1955).[10] This incompatibility of Val with PA is not a good reason to abandon the rule (Val-D), as Val is not intended to capture such a concept. Being characterized by the (Val-In) and (Val-D) rules, Val can be taken as characterizing a more general concept of validity, such as that of informal/absolute provability, proposed by Myhill (1960), which interprets

(Val-D) as saying that “all axioms of elementary arithmetic are true” (Myhill 1960, p.463).[11] That is, the predicate Val and $Prov_{\mathsf{PA}}$ stand for very different concepts. Although the former is required to capture the validity of the metarules of ST, which is a particular logical system, it still intends to capture general principles that govern informal validity, while the latter is a strictly local concept. Therefore, the incompatibility with the Second Incompleteness Theorem is not a good reason to abandon the (Val-D) rule in interpretating of Val as informal validity.

Last, but not least, Hlobil’s solution calls into question the diagnosis of the proponents of substructural approaches, according to which some substructural rules are the culprit in the derivation of paradoxes involving the naïve notion of validity. According to such a diagnosis, abandoning some structural rule(s) is necessary. However, Hlobil’s solution can be seen as requiring too much: to accept his solution, one has to abandon both the rule of Cut and the rule of detachment of Val. From a philosophical point of view, such a sacrifice may be considered unnecessary.

4. On the(se) solutions of validity paradoxes

Our discussion shows that the debate about the validity paradoxes is far from being solved. Although we have focused on only a few solutions to the problem of paradoxes, we argue that they somehow raise more general questions about the problem we are dealing with, and we also argue that a solution *a la* Ketland is more adequate for these paradoxes. The classical approaches must proceed by (i) restriction of some rules of Val or (ii) linguistic restrictions to avoid the inconsistencies caused by the interaction between the validity predicate and the self-reference devices. The linguistic blocking *a la* Skyrms can be criticized for being *ad hoc* because it prohibits diagonalization of the validity predicate. As we discussed before, it is possible to introduce diagonalization on the language, on the pain of relativizing the predicate Val to the level of the language in the hierarchy. In the first case, the restriction of (Val-In) to logical validities may be criticized because logical validity does not capture naïve validity. We will deal with this criticism below.

In the case of the non-classical solutions, the situation is not different. As we said in Section 3.2, many non-classical solutions to validity paradoxes based on structural non-classical logics are generally subjected to v-Curry. In the case of LP^{Val}, we saw that it can only deal with paradoxes if we adopt self-referential procedures weak enough to block the derivations of the abovementioned results. Otherwise, the dialetheist is forced to adopt hierarchies of languages. The problem with adopting hierarchies is that they contradict the purpose of the dialetheist approach to semantic paradoxes, which is to adopt semantically closed languages capable of dealing with their own semantic concepts without the risk of trivializing the theory. Moreover, if the weakening of self-referential procedure is the only viable way for a dialetheist solution of v-Curry that accommodates both (Val-In) and (Val-D), then the deductive power of the resultant semantic theory is significantly weak. By adopting a logic as an alternative to classical logic, it is reasonable to require that his/her basic logic has sufficient inferential power to prove basic arithmetical facts when the arithmetical language is considered. Of course, this excludes the majority of non-classical logics, because many of them are remarkably weak. LP is a clear example of weak logic because its conditional → does not validate modus ponens. For example, Picollo (2020) shows that some truth-theories based on paraconsistent logics fail in validating some important mathematical principles, such as the induction axiom.[12] So, if the proponent of the non-classical logic aims to give an alternative semantic theory to the classical one, the deductive weakness becomes a real challenge to her/him.

Substructural approaches, such as ST^{Val}, fail in internalizing their metainferences. In this respect, we can say that these theories are incomplete regarding the concept of validity, as they cannot demonstrate the validity of the metarules of the logical system that accommodates Val. Moreover, if they are the only theories capable of accommodating both (Val-In) and (Val-D) rules without the risk of the triviality imposed by Curry's paradox, then the failure to internalize their metarules shows that they are not successful, in part.

Another option is to restrict or abolish one of the principles (Val-In) or (Val-D). Modifying (Val-In) leads us to Ketland's proposal, which shows that classical logic is already sufficient for logical validity. The abandonment of the principle (Val-D) does not seem appropriate for the reasons we raised at the end of subsection 3.2.2.[13] Logical validity predicates *a la* Ketland seems to be one of the only options capable of maintaining prominent versions of (Val-In') and (Val-D) without adopting a hierarchy of languages while keeping the strong expressiveness of the semantic theory. In a classical theory, both principles, along with (Val-K), are enough to capture all the validities of FOL and to internalize the metarules of this logic.

An upholder of the naïve approach to validity may object that logical validity only explains the meaning of the logical constants. That is, she/he may argue that logical validity does not capture the intuitive notion of validity. To counter this objection, we can say that we do not know what intuitive validity is. Indeed, as many authors point out (*e.g.*, Smiley 1998, Smith 2011, Andrade-Lotero and Novaes 2012, Halbach 2020), intuitive validity is a rather confusing concept. We do not know what are the basic concepts that govern intuitive validity. It may carry aspects of necessary preservation of truth (Shapiro 2005, Griffiths 2014) and relevance (Smiley 1998), or it may be a purely deductive concept (Shapiro 2005).[14] As Smith (2011) points out, such questions about the constitutive concepts of intuitive validity lack determinate answers. Every systematization of this notion may lead to a different theory of consequence.

Second, a concept of validity that captures all deductive reasoning does not seem attainable. Such a concept assumes that there is an underlying logic to natural language and that intuitive validity is the notion underlying such logic. However, as Glanzberg (2015) argues, natural language is not closed under a consequence relation. That is, natural language does not have an underlying logic. He argues that logical and natural language are autonomous domains, although it is possible to draw connections between them.[15] So, such intuitive validity that intends to capture all the deductive practices does not seem to exist.

Obviously, there are informal notions of validity such as the following:

Definition 4.1. (Smith 2011) $Val'(\varphi)$ holds if φ is true in all cases in virtue of its logical form

The informality of Val' lies in the non-specification of the set of logical constants of the language.[16] Although informal, the notion expressed by Val' results from a conceptual sharpening and is not a pre-theoretical notion. As Smith shows, if φ is a first-order formula and the word *case* is understood in the lines of Tarskian semantics for FOL, this notion coincides with the formal notions of validity of FOL.[17] So, although there are informal notions of validity, we cannot expect that they capture the totality of deductive inferences.

Although logical validity, like Val of Definition 3.4 and Val' of Definition 4.1, captures an idealized fragment of natural language, it captures general principles about preservation of truth from a set of premises to the conclusion. So, the logical validity predicate is an adequate solution to the validity paradoxes because it preserves interesting versions of (Val-D) and (Val-In) without appealing to a hierarchy of languages.[18]

5. Conclusion

As the discussion in the preceding sections shows, both classical and non-classical solutions to validity paradoxes struggle with Diagonalization Lemma, so some restrictions must be made to block the derivation of v-Curry. Each solution comes with a cost. The cost of the non-classical solutions analyzed here is the loss of the deductive power of the theory. Pailos's solution, for example, provides a non-trivial approach to validity to the cost of the weak expressiveness of the theory. In the case of the substructural approach analyzed here, we saw that it suffers a kind of incompleteness because it fails to internalize its metarules.

Both classical solutions presented here do not describe naïve validity. While Ketland's approach deals with logical validity and is consistent with PA (if PA is consistent), Skyrms's approach deals with validity orthodoxly by appealing to a hierarchy of languages and severely restricted modal predicate. In this latter case, we saw that it is necessary to relativize the predicate Val to the level of the language in the hierarchy if diagonalization is a desirable device. So, a solution *a la* Ketland figures out as an adequate solution to the paradoxes of validity by maintaining a single predicate in the language of a theory such as PA. Moreover, the completeness theorem of FOL assures that Val captures all validities of the logical theory.

As Kennedy and Väänänen (2017) argue, although logical validity of FOL does not capture a pre-theoretical notion of validity (if it exists), it codifies a well-established mathematical practice, the truth-preservation reasoning. Of course, we could generalize this argument for non-classical logics. In the case of intuitionistic first-order logic, one could argue that this logic codifies the constructive mathematical practice, the preservation of constructive provability. In this relative perspective about validity, the properties of the predicate Val are relative to the particular system that accommodates it, and the completeness theorem will play a fundamental role since it will show that the predicate of validity at issue captures both semantic and proof-theoretical definitions of validity of the logical system. In what concerns the validity predicate of systems which are not complete, the problem is far from being simple, and it is left as an open problem.

References

Anderson, C. A. 1983. The paradox of the knower. *The Journal of Philosophy* **80**(6): 338–55.

Andrade-Lotero, E.; Novaes, C. D. 2012. Validity, the squeezing argument and alternative semantic systems: the case of aristotelian syllogistic. *Journal of philosophical logic* **41**(2): 387–418.

Asenjo, F. G. 1966. A calculus of antinomies. *Notre Dame Journal of Formal Logic* **7**(1): 103–5.

Barrio, E.; Rosenblatt, L.; Tajer, D. 2015. The logics of strict-tolerant logic. *Journal of Philosophical Logic* **44**(5): 551–71.

Barrio, E.; Rosenblatt, L.; Tajer, D. 2016. Capturing naive validity in the cut-free approach. *Synthese*: 1–17.

Beall, J.; Murzi, J. 2013. Two flavors of curry's paradox. *The Journal of Philosophy* **110**(3): 143–65.

Burgess, J. P. 1999. Which modal logic is the right one? *Notre Dame Journal of Formal Logic* **40**(1): 81–93.

Carnielli, W.; Coniglio, M. E.; Marcos, J. 2007. Logics of formal inconsistency. In: *Handbook of philosophical logic*, pp.1–93. Springer.

Cobreros, P.; Égré, P.; Ripley, D.; van Rooij, R. 2012. Tolerant, classical, strict. *Journal of Philosophical Logic* **41**(2): 347–85.

Cook, R. T. 2014. There is no paradox of logical validity. *Logica Universalis* **8**(3-4): 447–67.

Crocco, G. 2019. Informal and absolute proofs: some remarks from a Gödelian perspective. *Topoi* **38**(3): 561–75.

Dean, W. 2014. Montague's paradox, informal provability, and explicit modal logic. *Notre Dame Journal of Formal Logic* **55**(2): 157–96.

Dutilh Novaes, C.; French, R. 2018. Paradoxes and structural rules from a dialogical perspective. *Philosophical Issues* **28**(1): 129–58.

Égré, P. 2005. The knower paradox in the light of provability interpretations of modal logic. *Journal of Logic, Language and Information* **14**(1): 13–48.

Field, H. 2017. Disarming a paradox of validity. *Notre Dame Journal of Formal Logic* **58**(1): 1–19.

Glanzberg, M. 2015. Logical consequence and natural language. In: *Foundations of logical consequence*, pp.71–120.

Goodship, L. 1996. On dialethism. *Australasian Journal of Philosophy* **74**(1): 153–61.

Griffiths, O. 2014. Formal and informal consequence. *Thought: A Journal of Philosophy* **3**(1): 9–20.

Halbach, V. 2020. The substitutional analysis of logical consequence. *Noûs* **54**(2): 431–50.

Halldén, S. 1963. A pragmatic approach to modal theory. *Acta Philosophica Fennica* **16**: 53–63.

Hazen, A. 1984. Modality as many metalinguistic predicates. *Philosophical Studies* **46**(2): 271–7.

Hlobil, U. 2019. Faithfulness for naive validity. *Synthese* **196**(11): 4759–74.

Kennedy, J.; Väänänen, J. 2017. Squeezing arguments and strong logics. In: *15th International Congress of Logic, Methodology and Philosophy of Science*. College Publications.

Ketland, J. 2012. Validity as a primitive. *Analysis* **72**(3): 421–30.

Kreisel, G. 1967. Informal rigour and completeness proofs. In: *Studies in Logic and the Foundations of Mathematics*, volume 47, pp.138–86. Elsevier.

Leitgeb, H. 2009. On formal and informal provability. In: *New waves in philosophy of mathematics*, pp.263–99. Springer.

Löb, M. H. 1955. Solution of a problem of Leon Henkin 1. *The Journal of Symbolic Logic* **20**(2): 115–8.

MacFarlane, J. G. 2000. *What does it mean to say that logic is formal?*. University of Pittsburgh.

Meadows, T. 2014. Fixed points for consequence relations. *Logique et Analyse*: 333–57.

Montague, R. 1963. Syntactical treatments of modality, with corollaries on reflexion principles and finite axiomatizability. *Acta Philosophica Fennica*: **XVI**: 153–167.

Murzi, J.; Carrara, M. 2015. Paradox and logical revision. a short introduction. *Topoi* **34**(1): 7–14.

Murzi, J.; Rossi, L. 2017. Naïve validity. *Synthese*: 1–23.

Myhill, J. 1960. Some remarks on the notion of proof. *The Journal of Philosophy* **57**(14): 461–71.

Otte, R. 1982. Modality as a metalinguistic predicate. *Philosophical Studies* **41**(2): 153–9.

Pailos, F. M. 2020. Validity, dialetheism and self-reference. *Synthese* **197**(2): 773–92.

Picollo, L. 2020. Truth in a logic of formal inconsistency: How classical can it get? *Logic Journal of the IGPL* **28**(5): 771–806.

Priest, G. 1979. The logic of paradox. *Journal of Philosophical Logic* **8**(1): 219–41.

Priest, G. 2008. *An introduction to non-classical logic: From if to is*. Cambridge University Press.

Quine, W. V. O. 1966. Three grades of modal involvement (1953). In: Quine, W. V. O. (ed.) *The Ways of Paradox and Other Essays*. New York: Random House, p.158-176.
Ripley, D. 2012. Conservatively extending classical logic with transparent truth. *The Review of Symbolic Logic* **5**(2): 354–78.
Ripley, D. 2013. Paradoxes and failures of cut. *Australasian Journal of Philosophy* **91**(1): 139–64.
Rosenblatt, L. 2017. Naive validity, internalization, and substructural approaches to paradox. *Ergo, an Open Access Journal of Philosophy* **4**: 93–120
Sette, A. M. 1973. On the propositional calculus P1. *Mathematica Japonicae* **18**: 173–80.
Shapiro, L.; Beall, J. 2018. Curry's paradox. In: E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*. *Stanford University, summer 2018 edition*. https://plato.stanford.edu/entries/curry-paradox/
Shapiro, S. 2005. Logical Consequence, Proof Theory, and Model Theory. In: *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp.651–70. Oxford University Press.
Skyrms, B. 1978. An immaculate conception of modality or how to confuse use and mention. *The Journal of Philosophy* **75**(7): 368–87.
Smiley, T. 1998. Conceptions of consequence. *Routledge encyclopedia of philosophy*. London: Routledge.
Smith, P. 2011. Squeezing arguments. *Analysis* **71**(1): 22–30.
Stern, J. 2015. *Toward predicate approaches to modality*, volume 44. Springer.
Tarski, A. 1956. *Logic, semantics, metamathematics: papers from 1923 to 1938*. Oxford Clarendon Press.
Terzian, G. 2015. Norms of truth and logical revision. *Topoi* **34**(1): 15–23.
Varzi, A. C. 2002. On logical relativity. *Philosophical Issues* **12**(1): 197–219.
Weber, Z. 2014. Naive validity. *The Philosophical Quarterly* **64**(254): 99–114.
Whittle, B. 2004. Dialetheism, logical consequence and hierarchy. *Analysis* **64**(4): 318–26.
Zardini, E. 2013. Naive logical properties and structural properties. *The Journal of Philosophy* **110**(11): 633–44.

Notes

[1] We say minimal because some may defend (*e.g.*, (Halldén 1963, Burgess 1999) that validity also obeys $Val(\ulcorner\varphi \to \psi\urcorner) \to (Val(\ulcorner\varphi\urcorner) \to Val(\ulcorner\psi\urcorner))$ and $Val(\ulcorner\varphi\urcorner) \to Val(\ulcorner Val(\ulcorner\varphi\urcorner)\urcorner)$.

[2] The principle (Val-D) could be called (Val-T) due to its relation with the modal axiom $\Box\varphi \to \varphi$. But, we choose to follow the usual notation in the literature.

[3] To be honest, Montague's original result uses the arithmetic Q instead of PA. As we know, Q is a subsystem of PA, obtained by dropping the induction axiom from PA. Of course, since Q is a subsystem of PA, Montague's result holds for this latter system.

[4] These differences in the naming device are important and it will be explained during the presentation.

[5] Strictly speaking, Skyrms's work addresses Montague's theorem. However, as we will see, his solution is also immune to v-Curry.

[6] In this subsection, we will analyze a dialetheist proposal that is immune to v-Curry, but adopts a weaker self-referential procedure.

[7]It is a well-known fact that LP conditional does not validate modus ponens. Since $\varphi \rightarrow \psi$ is defined as $\neg\varphi \vee \psi$, then an easy counterexample refutes modus ponens in LP.

[8]If SSRP were introduced in LP^{Val}, it would be possible to define a sentence π that is intersubstitutable with $Val(\langle\pi\rangle, \langle\bot\rangle)$, and derive a stronger version of Theorem 2.3 that does not use the rules for the conditional. Such a proof can be found in Shapiro and Beall (2018).

[9]In this sense, bilateralism is a kind of logical inferentialism since the meaning of the logical constants is given by their sequent rules.

[10]Let $Prov_{\mathsf{PA}}$ be the provability predicate of PA. Roughly speaking, Löb's theorem states that the reflection principle $Prov_{\mathsf{PA}}(\ulcorner\varphi\urcorner) \rightarrow \varphi$ is provable in PA only in the trivial case that φ is already provable in PA. In Field (2017), we can find criticism against the rule (Val-D) in the light of Löb's theorem.

[11]This interpretation according to which (Val-D) is valid is compatible with Gödel's formalization of S4 as capturing a concept of informal probability (Leitgeb 2009, Dean 2014).

[12]In her work, she considers the logic LP enriched with a recovery operator $\circ$. The resulting logic is a *logic of formal inconsistency* in the sense of Carnielli et al. (2007).

[13]That at least one version of the (Val-In) rule is valid is a consensus in the literature.

[14]It is important to notice that this intuitive notion of validity is assumed to be purely deductive. If this notion is also intended to include inductive inferences, for example, the principle (Val-D) should not be considered valid because inductive validity is not necessarily truth-preserving.

[15]We are not saying that logical tools cannot be used to study inferences present in natural language. Indeed, several fruitful formalizations in philosophical logic and formal linguistics have been proposed to capture concepts present in natural language. However, this is not to say that there is a logic that captures all inferences present in the natural language.

[16]The division about what counts as a logical constant and what does not is an open and interesting philosophical problem, which we will not take an instance here. We invite the reader to check Varzi (2002) and Macfarlane (2000) for the discussion about such division.

[17]This is an example of Kreisel (1967)'s *informal rigour*.

[18]Logical validity is a local notion. One could say that Myhill's (1960) predicate of absolute provability is an example of a predicate encompassing all the deductive reasoning. However, as Crocco (2019) observes, Myhill's predicate is a highly formal notion, and its absolute character lies in the fact that we cannot say that our current mathematical theories capture an ultimate notion of mathematical proof.

Acknowledgments

The author acknowledges the financial support of Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación (Agencia I+D+I).