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# An overview of the Philosophy of Mathematics Education

## Um panorama da Filosofia da Educação Matemática

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**Abstract**

This paper focuses on the sub-field of study the philosophy of mathematics education from one perspective. The field is characterised in both narrow and broad terms, and from both bottom-up (questions and practices) and top-down perspectives (in terms of philosophy and its branches). From the bottom-up one can characterize the area in terms of questions, and I have asked: What are the aims and purposes of teaching and learning mathematics? What is mathematics? How does mathematics relate to society? What is learning mathematics? What is mathematics teaching? What is the status of mathematics education as knowledge field? I have characterized the sub-field using a 'top down' perspective using the branches of philosophy. Looking briefly into the contributions of ontology and metaphysics, aesthetics, epistemology and learning theory, social philosophy, ethics, and the research methodology of mathematics education reveals both how rich and deep the contributions of philosophy are to the theoretical foundations of our field of study. But even these two approaches leave many questions unanswered. For example: what are the responsibilities of mathematics and what is the responsibility of our own subfield, the philosophy of mathematics education? I conclude that the role of the philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research.

**Keywords:** Philosophy of Mathematics Education; Mathematics Education; Mathematics.

**Resumo**

Este artigo foca a filosofia da educação matemática de certa perspectiva, entendendo-a como uma subárea de estudo da educação matemática. Este campo de estudo pode ser caracterizado de maneira estreita ou abrangente, de duas perspectivas: uma de "baixo para cima" (por meio de questões originadas na prática) e de "cima para baixo" (a partir da própria filosofia e seus ramos). De "baixo para cima" pode-se caracterizá-la por meio das seguintes questões: quais são os objetivos e propósitos de ensinar e de aprender matemática? O que é matemática? Como a matemática se relaciona com a sociedade? O que é aprender matemática? O que é ensinar matemática? Qual é o status da educação matemática como campo de conhecimento? Esse campo também pode ser caracterizado usando uma perspectiva 'de cima para baixo' valendo-se da filosofia e seus ramos. Em um primeiro olhar, pode-se destacar possíveis contribuições valendo-se da ontologia e metafísica, da estética, da epistemologia, da teoria da aprendizagem, da filosofia social, da ética, que revelam a riqueza e profundidade das contribuições da filosofia para a metodologia de pesquisa em educação matemática, assim como para os fundamentos teóricos deste campo de estudo. Ainda assim, essas duas abordagens não esgotam a filosofia da educação matemática, deixando muitas questões não respondidas, tais como: quais são as responsabilidades dos matemáticos e qual é a responsabilidade de nossa própria subárea, a filosofia da educação matemática? Conclui-se que o papel da filosofia da educação matemática é analisar, questionar, desafiar e criticar as afirmações da prática, da política e da pesquisa em educação matemática.

**Palavras-chave**: Filosofia da Educação Matemática; Educação Matemática; Matemática.

## Introduction: What is the philosophy of mathematics education?

In the past 25 years or so the philosophy of mathematics education has emerged as a loosely defined area of research. It is primarily concerned with the philosophical aspects of mathematics education research. In this chapter my aim is to briefly map out the terrain, and to attempt to clarify the breadth and depths, especially as the question of what constitutes the philosophy of mathematics education is not without ambiguity and multiple answers.

In clarifying what the philosophy of mathematics education is, or what it might be, an immediate question arises. Is it *a* philosophy of mathematics education, or is it *the* philosophy of mathematics education? The preposition 'a' suggests an account offered that is one of several perspectives. In contrast, the definite article 'the' might imply the arrogation of definitiveness to the account. The latter is not what is intended here, for 'the' is meant to indicate a definite area of enquiry, a specific domain, within which one account or treatment is offered. So *the* philosophy of mathematics education need not be a dominant interpretation so much as an area of study, an area of investigation, and as here, an exploratory assay into this field.

The philosophy of mathematics education can be interpreted both narrowly and more widely. In the narrow sense the philosophy of some activity or domain can be understood as its aim or rationale. Understood in its simplest sense mathematics education is the practice or activity of teaching mathematics. So the narrowest sense of 'philosophy of mathematics education' concerns the aim or rationale behind the practice of teaching mathematics. The question of the purpose of teaching and learning mathematics is an important one, and must always be central to the philosophy of mathematics education. Learning is included here because it is inseparable from teaching. Although they can be conceived of separately, in practice an active teacher presupposes one or more learners. Only in pathological situations can one have teaching without learning, although of course the converse does not hold. Informal learning is often self directed and takes place without explicit teaching.

It should be remarked that the aims, goals, purposes, rationales, etc., for teaching mathematics do not exist in a vacuum. They belong to people, whether individuals or social groups (Ernest 1991). Since the teaching of mathematics is a widespread and highly organised social activity, its aims, goals, purposes, rationales, and so on, need to be related to social groups and society in general, while acknowledging that there are multiple and divergent aims and goals among different persons and groups. Aims are expressions of values, and thus the educational and social values of society or some part of it are implicated in this enquiry. In addition, the aims discussed so far are for the teaching of *mathematics*, so the aims and values implicated centrally concern mathematics and its role and purposes in education and society.

Thus a consideration of the narrow meaning of the philosophy of mathematics education immediately raises the issues of the teaching and learning of mathematics, the underlying aims and rationales for this activity, the roles of the teacher, learner, and mathematics in society and the underlying values of the relevant social groups. To a great extent this mirrors the issues arising from applying Schwab's (1961) four 'commonplaces of teaching' to mathematics. His commonplaces or basics of curriculum are the subject (mathematics), the learner of mathematics, the mathematics teacher, and the milieu of teaching, including the relationship of mathematics teaching and learning, and its aims, to society in general.

**Broader views of the philosophy of mathematics education**

There are broader interpretations of the philosophy of mathematics education that go beyond the aims, rationale and basis for teaching mathematics, and what that entails. Some of the expanded senses include:

1. Philosophy applied to or of mathematics education
2. Philosophy of mathematics applied to mathematics education or to education in general
3. Philosophy of education applied to mathematics education (BROWN 1995).

Each of these possible applications of philosophy to mathematics education represents a different focus, and might very well foreground different issues and problems. However, this analysis of applications of philosophy suggests that there are always substantive bodies of knowledge and activities connecting them in applications involved. In fact, philosophy, mathematics education and other domains of knowledge encompass processes of enquiry and practice, personal knowledge, and as well as published knowledge representations. They are not simply substantial entities in themselves, but complex relationships and interactions between persons, society, social structures, knowledge representations and communicative and other practices. In other words, the applications of philosophical processes, methods and critical modes of thought represent a further expanded sense of the philosophy of mathematics education, as follows.

4. The application of philosophical concepts or methods, such as a critical attitude to claims as well as detailed conceptual analyses of the concepts, theories, methodology or results of mathematics education research, and of mathematics itself (ERNEST, 1998; SKOVSMOSE, 1994).

Philosophy is about systematic analysis and the critical examination of fundamental problems. It involves the exercise of the mind and intellect, including thought, enquiry, reasoning and its results: judgements, conclusions beliefs and knowledge. There are many ways in which such processes as well as the substantive theories, concepts and results of past enquiry can be applied to and within mathematics education.

Why does philosophy matter? Why does theory in general matter? First, because it helps to structure research and inquiries in an intelligent and well grounded way, offering a secure basis for knowledge. It provides an overall structure slotting the results of cutting edge research into the hard-won body of accepted knowledge. But in addition, it enables people to see beyond the official stories about the world, about society, economics, education, mathematics, teaching and learning. It provides thinking tools for questioning the status quo, for seeing that 'what is' is not 'what has to be'; to see that the boundaries between the possible and impossible are not always where we are told they are. It enables commonly accepted notions to be probed, questioned and implicit assumptions, ideological distortions or unintended prejudices to be revealed and challenged. It also, most importantly, enables us to imagine alternatives. Just as literature can allow us to stand in other people's shoes and see the world through their eyes and imaginations, so too philosophy and theory can give people new 'pairs of glasses' through which to see the world and its institutional practices anew, including the practices of teaching and learning mathematics, as well as those of research in mathematics education.

At the very least, this analysis suggests that the philosophy of mathematics education should attend not only to the aims and purposes of the teaching and learning of mathematics (the narrow sense) or even just the philosophy of mathematics and its implications for educational practice. It suggests that we should look more widely for philosophical and theoretical tools for understanding all aspects of the teaching and learning of mathematics and its milieu. At the very least we need to look to the philosophy of Schwab's (1961) other commonplaces of teaching: the learner, the teacher, and the milieu or society. So we also have the philosophy of learning (learning mathematics in particular), the philosophy of teaching (mathematics) and the philosophy of the milieu or society (in the first instance with respect to mathematics and mathematics education) as further elements to examine, and then we must also consider the discipline of mathematics education as a knowledge field in itself.

Looking at each of these four commonplaces in turn, a number of questions can be posed as issues for the philosophy of mathematics education, understood broadly, to address, including the following.

**What is mathematics?**

What is mathematics, and how can its unique characteristics be accommodated within a philosophy? Can mathematics be accounted for both as a body of knowledge and a social domain of enquiry? Does this lead to tensions? What philosophies of mathematics have been developed? What features of mathematics do they pick out as significant? What is their significance for and impact on the teaching and learning of mathematics? What is the rationale for picking out certain elements of mathematics for schooling? How can and should mathematics be conceptualised and transformed for educational purposes? What educational and social values and goals are involved? Is mathematics itself value-laden or value-free? How do mathematicians work and create new mathematical knowledge? What are the methods, values and aesthetics of mathematicians? How does history of mathematics relate to the philosophy of mathematics? Is mathematics changing as new methods and information and communication technologies emerge?

This already begins to pose questions relating to the next area of enquiry.

**How does mathematics relate to society?**

How does mathematics education relate to society? What are the aims of mathematics education, i.e., the aims of teaching mathematics? Are these aims valid? Whose aims are they? For whom? Based on which values? Who gains and who loses in the process? How do social, cultural and historical contexts relate to mathematics, the aims of teaching, and the teaching and learning of mathematics? What values underpin different sets of aims? How does mathematics contribute to the overall goals of society and education? What is the role of the teaching and learning of mathematics in promoting or hindering social justice conceived in terms of gender, race, class, (dis)ability and critical citizenship? Are feminist and/or anti-racist mathematics education possible and what do they mean? What are their implications for the teaching and learning of mathematics? How is mathematics viewed by the public and perceived in different sectors of society? What impact does this have on education? What is the relationship between mathematics and society? What functions does it perform? Which of these functions are intended and visible? Which functions are unintended or invisible? To what extent do mathematical metaphors, such as the profit and loss balance sheet, or the spreadsheet permeate social thinking? What is their philosophical significance? To whom is mathematics accountable?

**What is learning (what is learning mathematics)?**

What assumptions, possibly implicit, underpin views of learning mathematics? Are these assumptions valid? Which epistemologies and learning theories are assumed? How can the social context of learning be accommodated in what are often individualistically-oriented and traditionally cognitive learning theories? What are the philosophical presuppositions of information processing, constructivist, social constructivist, enactivist, sociocultural and other theories of learning mathematics? Do these theories have any impact on classroom practice, and if so what? What elements of learning mathematics are valuable? How can they be and should they be assessed? What feedback loops do different forms of assessment create, impacting on the processes of teaching and learning of mathematics? How strong is the analogy between the assessment of the learning of mathematics and the warranting of mathematical knowledge? What is the role of the learner? What powers of the learner are or could be developed by learning mathematics? How does the identity of the learner change and develop through learning mathematics? Does learning mathematics impact on the whole person for good or for ill? To what degree do such beneficial/deleterious outcomes occur, under what learning conditions and how do these relate to the cultural context? Does learning mathematics impact differentially on students according to social and individual differences and identities, and if so how? How is the future mathematician and the future citizen formed through learning mathematics? How important are affective dimensions including emotions, attitudes, beliefs and values in learning mathematics? What is mathematical ability and how can it be fostered? Is the learning of mathematics accessible to all? How do cultural artefacts and technologies, including information and communication technologies, support, shape and foster the learning of mathematics? To what extent should student experiences of learning mathematics mirror or model the practices of research mathematicians? Is the learning of mathematics hierarchical, progressive or cumulative, as traditional theories tell us, and if so, to what extent?

**What is teaching (mathematics)?**

What theories and epistemologies underlie the teaching of mathematics? Are there any adequately articulated theories of teaching mathematics? What assumptions, possibly implicit, do mathematics teaching approaches rest on? Are these assumptions valid? What means are adopted to achieve the aims of mathematics education? Are the ends and means consistent? Can we uncover and explore different ideologies of education and mathematics education and their impact on teaching mathematics? What methods, resources and techniques are, have been, and might be, used in the teaching of mathematics? Which of these have been helpful and under what circumstances and conditions? What theories underpin the use of different information and communication technologies in teaching mathematics? What sets of values do these technologies bring with them, both intended and unintended? Is there a philosophy of technology that enables us to understand the mediating roles of tools between humans and the world? What is it to know mathematics in a way that fulfils the aims of teaching mathematics? How can the teaching and learning of mathematics be evaluated and assessed? What is the role of the teacher? What range of roles is possible in the intermediary relation of the teacher between mathematics and the learner? What are the ethical, social and epistemological boundaries for the actions of the teacher? What mathematical knowledge, skills and processes does the teacher need or utilise? What is the range of mathematics-related beliefs, attitudes and personal philosophies held by teachers? How do these attitudes, beliefs and personal philosophies impact on mathematics teaching practices? How should mathematics teachers be educated? What is the difference between educating, training and developing mathematics teachers? What is (or should be) the role of research in mathematics teaching and the education of mathematics teachers?

One further set of questions for the philosophy of mathematics education goes beyond Schwab's four commonplaces of teaching, which applied here are primarily about the nature of the mathematics curriculum. This further set concerns the status of mathematics education in itself as a field of knowledge, and coming to know within it.

**What is the status of mathematics education as knowledge field?**

What is the basis of mathematics education as a field of knowledge? Is mathematics education a discipline, a field of enquiry, an interdisciplinary area, a domain of extra-disciplinary applications, or what? Is it a science, social science, art or humanity, or none or all of these? What is its relationship with other disciplines such as philosophy, mathematics, sociology, psychology, linguistics, anthropology, etc.? How do we come to know in mathematics education? What is the basis for knowledge claims in research in mathematics education? What research methods and methodologies are employed and what is their philosophical basis and status? How does the mathematics education research community judge knowledge claims? What standards are applied? How do these relate to the standards used in research in general education, social sciences, humanities, arts, mathematics, the physical sciences and applied sciences such as medicine, engineering and technology? What is the role and function of the researcher in mathematics education? Should we just focus on technical aspects of improving the teaching and learning of mathematics, or are we also public intellectuals whose responsibilities include critiquing mathematics and society? What is the status of theories in mathematics education? Do we appropriate theories and concepts from other disciplines or 'grow our own'? Which is better? What impact on mathematics education have modern developments in philosophy had, including phenomenology, critical theory, post-structuralism, post-modernism, Hermeneutics, semiotics, linguistic philosophy, etc.? What is the impact of research in mathematics education on other disciplines? What do adjacent STEM education subjects (science, technology, engineering and mathematics education) have in common, and how do they differ? Can the philosophy of mathematics education have any impact on the practices of teaching and learning of mathematics, on research in mathematics education, or on other disciplines? What is the status of the philosophy of mathematics education itself? How central is mathematics to research in mathematics education? Does mathematics education have an adequate and suitable philosophy of technology in order to accommodate the deep issues raised by information and communication technologies?

These five sets of questions encompass, in my view, much of what is important for the philosophy of mathematics education to consider and explore. These sets are not wholly discrete, as various areas of overlap reveal. Many of the questions are not essentially philosophical, in that they can also be addressed and explored in ways that foreground other disciplinary perspectives, such as sociology and psychology. However, when such questions are approached philosophically, they become part of the business of the philosophy of mathematics education. Also, if there were a move to exclude any of these questions right from the outset without considering them it would risk adopting or promoting a particular philosophical position, a particular ideology or indeed a slanted philosophy of mathematics education. Lastly, perhaps more so than philosophy, sociology or psychology, mathematics education is a multi- or inter-disciplinary area of study, so that it is perhaps the most appropriate area where all of these questions and sub-questions can be explored together from a philosophical perspective.

## A 'top down' analysis of the philosophy of mathematics education

The questions listed above can be taken to represent a 'bottom up' introduction to the philosophy of mathematics education, because they start with, interrogate and problematise the practices of teaching and learning mathematics and related issues from a non-theoretical perspective. In contrast, a 'top down' approach uses the abstract branches of philosophy to provide the conceptual framework for analysis. Thus it considers research and theory in mathematics education according to whether it draws on metaphysics and ontology, epistemology, social and political philosophy, ethics, methodology, aesthetics or other branches of philosophy.

Ontology and metaphysics have as yet been little applied in mathematics education research (ERNEST, 2012). Work drawing on aesthetics is still in its infancy (ERNEST, 2013, ERNEST, 2015a, SINCLAIR, 2008). However extensive uses of epistemology and learning theory, social and political philosophy, ethics and methodology can be found in mathematics education research.

**Ontology and metaphysics**

Ontology is that part of metaphysics that studies the nature and conditions of existence and being in itself. Although as yet little applied in mathematics education research ontology raises two immediate problem areas including first mathematical objects and second human beings (ERNEST, 2012). Platonism, which concerns the first of these issues, has been a dominant philosophy of mathematics for over two thousand years. It is the view that mathematical objects exist independently of the physical world in some ideal realm. However, there have been longstanding disputes in this area between Platonists or realists, and conceptualists and nominalists. Although sociologists and social constructivists have challenged Platonism it is only recently that mainstream philosophy has countenanced the idea that there is a fully existent social reality (SEARLE, 1995) and that mathematical objects are part of this social reality rather than some other reality (COLE, 2013; HERSH, 1997). Such thinking will doubtless also have consequences for the philosophy of technology and the status of the virtual realities brought into being by information and communication technologies, as well as the philosophy of mathematics. All I will signal here is that this is a controversial but burgeoning area of inquiry of great significance for our field. For it is largely through the teaching and learning of mathematics that learners meet, develop relationships with, and come to believe in the reality of mathematical objects and the certainty of mathematical knowledge (ERNEST, 2015b).

The nature of human being is another deep question that has implications for the teaching and learning of mathematics. What is the deep nature, the “non-essential essence” of learners, teachers and persons in general presupposed by teaching, learning and research in mathematics education? Of course such concerns also have immediate ethical consequences, but what do we add to mathematics education research by focussing on and clarifying these deep ontological issues? What new researchable projects are suggested and brought within our reach?

**Aesthetics**

Work drawing on aesthetics is still in its infancy but is growing (INGLIS; ABERDEIN, 2015; ERNEST, 2013; ERNEST, 2015a; SINCLAIR; 2008). Aesthetics has been associated with mathematics since the time of Plato, but what does the theoretical focus on aesthetics in research in mathematics education add beyond letting learners experience some of the beauty of mathematics? It is a commonplace that some mathematical proofs and some mathematical objects and theories are beautiful. But why are there such divergences of opinion between those who exalt the sublime beauty of mathematics and those who fail to see any beauty at all in mathematics? Are the differences of opinion intrinsic or are they down to the unique personal learning trajectories of some individuals? What can a focus on beauty in mathematics and its teaching and learning add to research and classroom teaching? Since the experience of beauty is usually associated with interest, admiration and other positive attitudes, can these be harnessed to improve learning experiences and overall engagement with mathematics?

**Epistemology**

Epistemology concerns theories of knowledge and can be taken to include both the nature of mathematical knowledge, including its means of verification, and the processes of coming to know, or learning. Thus some of the questions posed above (What is mathematics?) and (What is learning mathematics?) fall under this heading. There is a literature exploring the relationships between epistemologies of mathematics and mathematics education (ERNEST, 1994, 1998, 1999; SIERPINSKA; LERMAN, 1997). This literature provides frameworks for examining some of the main epistemological questions concerning truth, meaning and certainty, and the different ways they can be interpreted for our field. It surveys a range of epistemologies including the contexts of justification and discovery, foundational and non-foundational perspectives on mathematics, critical, genetic, social and cultural epistemologies, and epistemologies of meaning. Looking within mathematics education a number of epistemological controversies can be mapped out including the subjective-objective character of mathematical knowledge; the role in cognition of social and cultural context; the transfer of knowledge and the transfer of learning from one social context to another; relations between language and knowledge; and tensions between the major tenets of constructivism, sociocultural views, interactionism and French Didactique, from an epistemological perspective. Relationships between epistemology and a theory of instruction, especially in regard to didactic principles, can also considered, thus addressing the question 'What is teaching in mathematics?', since teaching is the deliberate attempt to direct and foster learning.

Work by sociologists on epistemology and the sociology of knowledge, including that of Bloor (1991) and Bernstein (1999), have impacted on our field through foregrounding sociological theories of knowledge. Even more radical impacts stem from the post-structuralism of Foucault (1980) and others, and the post-modernism of Lyotard (1984) and Derrida (1978). However, the impact of their theories cannot be confined solely to epistemology since they question and critique the traditional divisions of philosophy and knowledge. Their theories and accounts serve to destabilize traditional conceptions of the fixity of knowledge and the definiteness of concepts. There is a growing body of literature and theory that applies the insights of these recent social theories, if I can term them that, to mathematics education research (LLEWELLYN, 2010; HOSSAIN *et al.,* 2013).

**Learning theory**

Although the natural home of learning theory is in the domain of cognitive psychology, much has been made of their epistemological assumptions and implications within mathematics education research. Many tyro researchers in our field cut their philosophical teeth on the controversy over radical constructivism. The heated public debates at Psychology of Mathematics Education (PME) Conference no. 7 in Montreal in 1987 between Ernst von Glasersfeld, Jeremy Kilpatrick and David Wheeler foregrounded these issues for the international mathematics education research community. Striking and important philosophical differences can be found between the leading learning theories in our field. Although the controversy has calmed down since those first heady days it remains understood that there are major differences in the philosophical presuppositions of information-processing, constructivist, social constructivist, enactivist, and sociocultural theories of learning mathematics. These are primarily epistemological differences, although proponents and critics of the various theories also bring ontological, ethical, social and methodological analyses and reasoning into their arguments.

**Social and Political Philosophy**

Social and Political Philosophy is harder to pin down than some of the other branches of philosophy since the emergence of sociology which has contested and colonised some of its terrain. But there is a long and honourable tradition of political and social philosophy going back to Plato's Republic. In it Plato suggests how a society might best be organised on philosophical lines. In addition Plato also enunciates what might be termed the first philosophy of mathematics education. He argues that the learning of mathematics not only prepares philosophers to be future rulers, and provides important practical knowledge for builders, traders and soldiers, but more importantly also introduces its students to truth, the art of reasoning, and also to the key ideas of ethics. Such knowledge, he argues, is necessary at all levels in society, especially the top. As is well known, his academy, probably the first university in the world, and certainly one of the longest enduring, required that all who enter be versed in geometry.

The political philosopher *par excellence* of modern times is Karl Marx. His social and political analysis is primarily based on a critique of the economic structure of society and the role of capital. However, there is a strong ethical dimension to his work because his critique focuses on the exploitation of one social class by another and his outrage at this is palpable. Several schools of philosophy have built on Marx's insights including the Frankfurt School of critical theory, post-structuralist philosophy including Foucault (1980), Pierre Bourdieu's social theory (e.g., BOURDIEU; PASSERON, 1977). All of these are extensively used in mathematics education research. All of the movements mentioned are continental (primarily French and German) but the most widely cited non-continental social thinker in mathematics education research, Basil Bernstein, does not base his work on Marx. All of these named scholars or movements, whether primarily social or philosophical, have been used to make important philosophical contributions within mathematics education research.

Some of the other contributions of social and political theorising in mathematics education research have been critiques of individualistic conceptions of learning, persons and knowledge and the use of the social construct of 'identity' as a unit of analysis in researching and teaching mathematics (LERMAN, 2012).

**Ethics**

Ethics enters into mathematics education research in a number of ways including a concern with values, with social justice and equity approaches, and through the ethics of research methodology. Several authors have argued that despite its traditional value-free absolutist image mathematics is value laden (ERNEST, 2013). Others draw on Paulo Freire's (1972) emancipatory philosophy, again based on Marx, to argue that learning mathematics can be a revolutionary activity and should be emancipatory and empowering through forstering a critical citizenry. Prominent in taking these ideas forward, although not necessarily drawing on Freire, are the movements of critical mathematics education (SKOVSMOSE, 1994) and ethnomathematics (D'AMBROSIO, 2007). Because of the prominent role of ethics in these movements I mention them here, but their powerful social critiques could just as easily have been included under the heading of social and political philosophy, especially since critical mathematics education explicitly draws on the Frankfurt school.

Another dominant strand of ethics-driven research in mathematics education concerns social justice and its deficiencies in the education of special groups such as females, ethnic minorities, students with disabilities, special needs students, second language learners, students of lower socio-economic status, and so on. These righteous concerns have spawned a vast literature over the past forty years with many thousands of publications as well as dedicated conferences and research groups. Once again much of such research could also be labelled social and political but that which has a an overt philosophical dimension often predominantly focuses on the ethics of exclusion or disadvantage, so it fits here.

**Methodology**

Lastly, an area of mathematics education research in which philosophical issues are influential and overtly utilitised is that of research methodology. Serious research in our field, whether in the form of smaller projects such as doctoral investigations, or larger funded research projects, is expected to address the philosophical issues in research methodology. Beyond techniques and methods, research methodologies are expected to have a sound basis with explicit awareness and treatment of the ontological and epistemological assumptions underpinning the study, not to mention its ethics. Non-empirical research, being conceptual or philosophical, is even more required to be on top of its philosophical assumptions. Mathematics education is an interdisciplinary field of study straddling the sciences, social sciences, humanities and perhaps even the arts, so it is not surprising that a wide range of research methodologies and paradigms are employed in research. Indeed this diversity of research paradigms, approaches and methodologies is one of the great strengths of our filed. Nevertheless, philosophical justification is needed for the appropriateness of whatever research approach is chosen and employed, as well as for the validity and trustworthiness of the knowledge produced.

**Conceptual Analysis**

In addition to the contributions of the substantive branches of philosophy to mathematics education, there are also benefits to be gained from applying philosophical styles of thinking in our research. For example, many of the constructs we utilise need careful conceptual analysis and critique. I have in mind such widely used ideas as understanding, development, progress, progressivism, mathematical ability, nature/natural, values, objectivity/subjectivity, identity, working like a mathematician, learning, discovery learning, problem solving (including pure, applied, 'real' and 'authentic' problems), teaching, assessment, mathematics, knowledge, sex/gender, special needs in mathematics, multiculturalism/antiracism, ethnomathematics, context, both social and task-related, and so on.

Deconstructing some of these ideas/terms might seem 'old hat', but even an apparently everyday idea like understanding contains hidden assumptions and pitfalls. First of all, it is based on the peculiar metaphor of 'standing under'. In what way does this capture its meaning? Synonyms like 'grasping', 'getting a handle on' or 'seeing' it are all based familiarity through a sensory encounter with meaning, and on being able to control or possess it ('getting it'). Thus these metaphors presuppose a static 'banking' model, interpreting understanding as the acquisition, ownership or possession of knowledge (SFARD, 1998). But secondly, there is an ideological assumption that understanding a concept or skill is better, deeper and more valuable than simply being able to use or perform it successfully. Skemp (1976) distinguished 'relational understanding' from 'instrumental understanding', and posited the superiority of the former. However his co-originator of the distinction Stieg Mellin-Olsen (1979) used it to distinguish the modes of thinking of academic students from that of apprentices, thus bringing in a social context and even a social class dimension to the distinction, and imposing less of an implicit and gratuitous valuation. If we want to assert the superiority of 'relational understanding' over 'instrumental understanding' it needs to be done on the basis of a reasoned argument, and not taken for granted as obvious. Skemp's own argument was based on the psychology of schemata, based on Piagetian theories, but this have been challenged by a number of alternate theories of learning including socio-cultural theory and social constructivism, drawing on Vygotsky's (1986) theory of learning. According to Vygotsky knowledge is not something that the learner possesses but is a competence inferred from the learner's manifested ability to complete a task, either unaided, or, with the help of a more capable other, in what is termed the learner's zone of proximal development. Given current challenges to the underlying theories of learning, the assumption that relational understanding is superior stands in need of justification.

Some scholars have challenged the unquestioned pre-eminence of relational understanding. Hossain *et al.* (2013) question the accepted good of the related notion of 'understanding mathematics in-depth' because, as they show, its role in the identity work of some student-teachers is troubling to them.[1] For example, one student teacher with the pseudonym Lola experiences a conflict between the imposed good of relational understanding, when studying in England, and her own success within the norms of instrumental understanding that she internalized in her Nigerian upbringing (ERNEST, 2014). Others have challenged the uncritical promotion of understanding within the mathematics education community because of its incoherence. Llewellyn (2010) questions 'understanding' partly because of slippage in the use of the term so that it encompasses both its relational and instrumental forms. However, her deeper critique is that in use it carries with a whole host of problematic assumptions about who can own 'understanding' in terms of ability, gender, race, class.

> Understanding is produced as hierarchical, particularly in relation to gender, social class and ability. It belongs to the privileged few, the 'naturally' able, which are often boys (another unhelpful and unnecessary classification). To suggest that girls have a 'quest for understanding' is over simplistic and gendered and in the first instance we should unpack how each version of understanding is constructed. … Finally I suggest that student teachers do not produce understanding as cognitive; the child is not an automaton who performs as the government text prescribes. Pupils and understanding are tied up with notions such as gender, confidence and emotions (LLEWELLYN, 2010, p. 355-356).

What this example shows is that a widely presupposed good in the discourse of mathematics education, the concept of understanding, is a worthwhile target of philosophical analysis and critique. Although such analysis does not mean that we have to abandon the concept, it does mean that we need to be aware of the penumbra of meanings revealed and aporias unleashed through its deconstruction. We need to use the term with caution and precision, clarifying or sidestepping its troubling connotations and implications. Thus the philosophy of mathematics education, as well as offering valuable overarching and synoptic views and explanations of our field, also serves as an under-labourer.[2] It can clear the conceptual landscape of unnoticed obstacles and perform the hygienic function of targeting, inoculating and neutralizing potentially toxic ideas circulating, like viruses, in our discourse.

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[1] In later work Skemp (1982) refers to instrumental understanding as 'surface' and relational understanding as 'deep' understanding, thus prefiguring the depth metaphor in the more recently coined term 'understanding mathematics in-depth'.

[2] "[I]t is ambition enough to be employed as an under-labourer in clearing the ground a little, and removing some of the rubbish that lies in the way to knowledge" (Locke 1975, p. 10).

## Conclusion

In this chapter I have attempted to outline the sub-field of study the philosophy of mathematics education from my perspective. I have characterized this in both narrow and broad terms, and from both the bottom-up and top-down perspectives. From the bottom-up perspective one can characterize the area in terms of questions, and I have asked: What are the aims and purposes of teaching and learning mathematics? What is mathematics? How does mathematics relate to society? What is learning mathematics? What is mathematics teaching? What is the status of mathematics education as knowledge field? I have also characterized the sub-field using a 'top down' perspective using the branches of philosophy. Looking briefly into the contributions of ontology and metaphysics, aesthetics, epistemology and learning theory, social philosophy, ethics, and the research methodology of mathematics education reveals both how rich and deep the contributions of philosophy are to the theoretical foundations of our field of study.

But this little assay into the topic is just the beginning, for there are many more unanswered questions. For example: what are the overall responsibilities of mathematics education as an overall field of study and practice, and what is the responsibility of our own subfield, the philosophy of mathematics education? What are the responsibilities of mathematics education researchers? Does this depend on our philosophical stances, whether we see ourselves as critical public intellectuals or as functional academics probing deeper into narrow specialisms?

Philosophy emerged from the dialectics of the ancient Greeks where commonplace beliefs and unanalysed concepts were interrogated and scrutinised, where the role of the rulers was questioned and challenged through speaking truth to power. Thus the role of the philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research. Our job is to unearth hidden assumptions and presuppositions, and by making them overt and visible, to enable researchers and practitioners to boldly go beyond their own self-imposed limits, beyond the unquestioned conceptual boundaries installed by the discourse of our field, to work towards realizing their own dreams, visions and ideals.

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